Note on a Generalization of Roth's Theorem

József Solymosi

Abstract

We give a simple proof that for sufficiently large N, every subset of $[N]^2$ of size at least δN^2 contains three points of the form $\{(a,b),(a+d,b),(a,b+d)\}$.

In this note we give a simple proof for a theorem of Ajtai and Szemerédi [1]. In their proof Ajtai and Szemerédi used and iterated Szemerédi's theorem about long arithmetic progressions in dense sets of integers [8]. A more general theorem of Fürstenberg and Katznelson also implies Theorem 1, but does not give bound on N as it uses ergodic theory [2]. After improving the bound in Szemerédi's theorem, Gowers asked for a quantitative proof of Theorem 1 [3, 4].

Theorem 1 *(Ajtai-Szemerédi) For any real number $\delta > 0$ there is a natural number N_0 such that for $N > N_0$ every subset of $[N]^2$ of size at least δN^2 contains a triple of the form $\{(a,b),(a+d,b),(a,b+d)\}$ for some integer $d \neq 0$.*

The key of the proof is a lemma of Ruzsa and Szemerédi [7]. A subgraph of a graph G is a *matching* if every vertex has degree one. A matching M is an *induced matching* if there are no other edges of G between the vertices of M.

Lemma 2 *(Ruzsa-Szemerédi) If G_n is the union of n induced matchings, then $e(G_n) = o(n^2)$.*

The lemma, with a simple proof deduced from Szemerédi's Regularity Lemma, can be also found in a survey paper of Komlós and Simonovits [5].

Proof of Theorem 1: Let S be a subset of the grid $[N]^2$ of size at least δN^2. We refer to a point of the grid with its coordinates, which are pairs $(i,j); i,j \in \{1,2,\dots,N\}$. Let us define a bipartite graph $G(A,B)$ with vertex sets $A = \{v_1,\dots,v_N\}$ and $B = \{w_1,\dots,w_N\}$. Two vertices v_i and w_j are connected by an edge iff $(i,j) \in S$ (see Fig. 1).

Let us partition the edges of G according to their length, $(v_i,w_j) \sim (v_l,w_m)$ iff $i+j = l+m$. Every partition class is a matching, so we can apply Lemma 2 to G. If N is large enough, then at least one matching is not induced. A triple

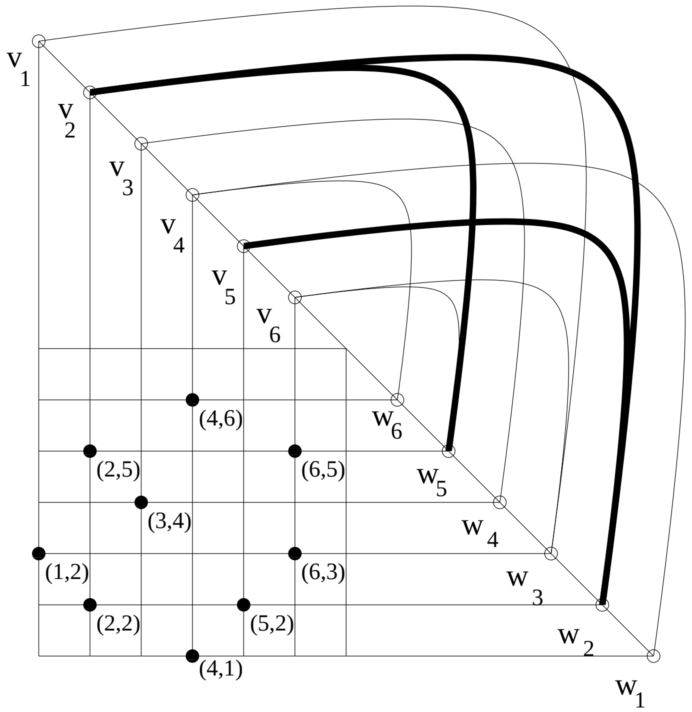

Figure 1: Converting points into edges

of edges $(v_i, w_m), (v_i, w_j), (v_l, w_m)$ such that $(v_i, w_j) \sim (v_l, w_m)$ guarantees a triple in S, $\{(a,b), (a+d,b), (a,b+d)\}$ (see bold edges in Fig.1). $\square$

The only known proof of Lemma 2 uses Szemerédi's Regularity Lemma [8], so while the proof is quantitative, it gives a tower-type bound on $N_0 = N_0(\delta^{-1})$. It would be very important to find another, maybe analytical proof for Lemma 2 to get a better bound.

References

[1] M. Ajtai and E. Szemerédi, Sets of Lattice Points That Form No Squares. *Studia Scientiarum Mathematicarum Hungarica.* **9** (1974), 9–11.

[2] H. Fürstenberg and Y. Katznelson, A density version of the Hales-Jewett theorem. *J.d'Analyse Math.***57** (1991), 64–119.

[3] W.T. Gowers, A new proof of Szemerédi's theorem. *GAFA, Geom. Funct. Anal.* **11** (2001) 465–588.

[4] W.T. Gowers, Rough structure and classification. *GAFA, Geom. Funct. Anal.* Special Volume - GAFA2000 "Visions in Mathematics", Tel Aviv, 1999. Part I, 79–117.

[5] J. Komlós and M. Simonovits, Szemerédi's Regularity Lemma and its applications in graph theory. in: Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), 295–352, Bolyai Soc. Math. Stud., 2, János Bolyai Math. Soc., Budapest, 1996.

[6] K.F. Roth On certain sets of integers, *J.London Math. Soc.***28** (1953), 245–252.

[7] I.Z. Ruzsa and E. Szemerédi, Triple systems with no six points carrying three triangles. in: Colloquia Mathematica Societatis János Bolyai, 18. Combinatorics, Keszthely (Hungary), 1976, 939–945.

[8] E. Szemerédi, On sets of integers containing no k elements in arithmetical progression. *Acta Arithmetica* **27** (1975), 199–245.

About Author

József Solymosi is at the Department of Mathematics, University of California, San Diego, La Jolla, CA 92093-0112, USA. *solymosi@math.ucsd.edu*.

Acknowledgments

I thank Timothy Gowers and Tibor Szabó for useful discussions. Work on this paper has been supported by the Berlin-Zürich European Graduate Program "Combinatorics, Geometry, and Computation" and by the Computer and Automation Research Institute, Hungarian Academy of Sciences.