

Corners in Cartesian products

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This note is an illustration of the density-increment method used in the proof of the density Hales-Jewett theorem for $k = 3$. (Polymath project [2]) I will repeat the argument applying it to a problem which is easier than DHJ. In the last section I will describe the proof of the density Hales-Jewett theorem for $k = 3$. The results stated here are direct interpretations of the project's results, no originality is claimed.

1 Preliminaries

A finite set of real numbers H is called *Hilbert-cube* (or *cube* in short) if there is a set of reals x_0, x_1, \dots, x_d such that

cubes

$$H = H(x_0, x_1, \dots, x_d) = \left\{ x_0 + \sum_{i \in I} x_i \mid I \subset [1, 2, \dots, d] \right\}.$$

Number d is the dimension of the cube. In this note we will always suppose that the generators, x_1, \dots, x_d , are independent in the sense that all elements of the cube are distinct, $|H| = 2^d$.

We are going to use the following notation (from the Polymath1 Wiki [2]): An m -dimensional combinatorial subspace in $[k]^n$ is obtained by taking m disjoint subsets W_1, \dots, W_m of $[n]$, fixing the values of all coordinates outside these subsets, and taking all points that equal the fixed values outside the W_i and are constant on each W_i .

subspace

The next theorem is due to Gunderson, Rödl, and Sidorenko.

Theorem 1.1 ([4]) *For any integer $m > 2$ there is a constant $c_m > 0$ such that if $\mathcal{F} \subset [2]^n$ and $|\mathcal{F}| \geq c_m n^{-1/2^m} 2^n$ then \mathcal{F} contains an m -dimensional combinatorial subspace.*

Let us denote the maximum size of a family $\mathcal{F} \subset [2]^n$ which does not contain an m -dimensional combinatorial subspace by $M(m, n)$.

$M(m, n)$

Theorem 1.1 is a simple but useful statement, which we are going to use twice. Our first application is to show that large subsets of cubes contain large cubes.

Corollary 1.2 *Let S be a subset of an n -dimensional cube $H(x_0, x_1, \dots, x_n)$. If $|S| > M(m, n)$ then S contains an m -dimensional cube $H'(x'_0, x'_1, \dots, x'_m)$.*

Proof: Every element of S is uniquely represented by a subset of $[1, n]$. By Theorem 1.1 there is an m dimensional combinatorial subspace among these sets. If W_0 denotes the set of 1-s outside of W_i -s in this combinatorial subspace (see the notations above), then H' is given by $x'_0 = x_0 + \sum_{i \in W_0} x_i$ and $x'_j = \sum_{i \in W_j} x_i$ for $1 \leq j \leq m$. ■

2 Corners

The result we are going to prove is a generalization of the “Corners Theorem” by Ajtai and Szemerédi.

Theorem 2.1 ([3]) *If N is sufficiently large depending on δ , then any δ -dense subset of $[N]^2$ must contain a “corner”, three points with coordinates $(x, y), (x + r, y), (x, y + r)$ with $r \neq 0$.*

The main result of this note is the following:

Theorem 2.2 *If the dimension of a cube H is sufficiently large depending on $\delta > 0$, then any δ -dense subset of the Cartesian product $H \times H$ must contain a “corner”, three points having coordinates $(x, y), (x + r, y), (x, y + r)$ with $r \neq 0$.*

The proof is based on a density increment method. We will show that if a dense subset is corner-free, then it is very dense in a Cartesian product of a small (but not very small) cube. The larger density then guarantees a corner there anyways. The key of the proof is a technical lemma which might be useful for other applications as well. Before we start the proof, let me give a few remarks. With Ron Graham we gave a simple combinatorial proof to the coloring version of the theorem in [5]: There is an absolute constant $c > 0$ that if H is d dimensional then any coloring of the Cartesian product by $c \log d$ colors results in a monochromatic corner. There are some interesting equivalent statements. We will state two of them for the density version. The following statements are corollaries of Theorem 2.2. (Please refer to [5] for the details.)

Corollary 2.3 *Let G be a geometric graph on the d dimensional cube in \mathbb{R}^d , with straight line segments as edges between the vertices of the cube. If G is dense and d is large enough then there is a selfcrossing path of length three. (something like this: \times)*

The next statement is a weak density Hales-Jewett theorem.

Corollary 2.4 *For every $c > 0$ there is an $n = n_0(c)$, with the following property. Any c -dense subset of $\{0, 1, 2, 3\}^n$ with $n > n_0$, contains a set of 3 points of the form:*

$$\begin{aligned} &(\dots, a, \dots, 0, \dots, b, \dots, 3, \dots, 0, \dots, c, \dots, 3, \dots, d, \dots) \\ &(\dots, a, \dots, 1, \dots, b, \dots, 2, \dots, 1, \dots, c, \dots, 2, \dots, d, \dots) \\ &(\dots, a, \dots, 2, \dots, b, \dots, 1, \dots, 2, \dots, c, \dots, 1, \dots, d, \dots) \end{aligned}$$

In other words, every column is either *constant*, *increasing* from 0, or *decreasing* from 3.

2.1 Proof

Let us suppose that Theorem 2.2 is false; there is a $c > 0$ that there is a c -dense subset of $H \times H$, without a corner for arbitrarily large H cubes. Let \bar{c} denote the lim sup of c -s with this property. We will show that if S is corner-free and it is a c -dense subset of $H \times H$ (where H is an n dimensional cube), then a subset of S is $c + \delta$ -dense on a product $H' \times H'$, where H' is an m dimensional cube. If we can choose m arbitrarily large and that $c + \delta > \bar{c}$, where δ is independent of m , then the proof of Theorem 2.2 is complete.

The next statements show that large Cartesian products in products of cubes correlate to smaller products of cubes. I think that Lemma 2.6 is the key for the proof of Theorem 2.2, other parts of the proof are quite straightforward.

Lemma 2.5 *For any reals $\varepsilon > 0$, $\alpha > 0$ and integer $m > 2$ there is a threshold n_0 that the following holds; if A and B are subsets of an n dimensional cube H , $n \geq n_0$, $|A| \geq |B| \geq \alpha|H|$, and S is a δ -dense subset of $A \times B$, then there is an m dimensional cube H' and a real number ξ that S is at least $(\delta - \varepsilon)$ -dense on $H' \times (H' + \xi)$.*

The lemma above follows directly from the next one which states that $A \times B$ has an almost complete tiling by Cartesian products of m dimensional cubes. We say that $A \times B$ has an ε -complete tiling by a family of two-dimensional point sets $\mathcal{P} = \{P_1, \dots, P_K\}$ if there is a $C \subset A \times B$ that $|A \times B \setminus C| \leq \varepsilon|A||B|$ and C is a disjoint union of translates of sets from \mathcal{P} .

Lemma 2.6 *For any real $\varepsilon > 0$ and integer $m > 2$ there is a threshold n_0 and an integer $K = K(\varepsilon, m)$ that the following holds; if A and B are subsets of an n dimensional cube $H(x_0, x_1, \dots, x_n)$, $n \geq n_0$, $|A| \geq |B| \geq c2^n$, then there are K (not necessary distinct) m dimensional cubes $H^{(1)}, \dots, H^{(K)}$ that $A \times B$ has an ε -complete tiling by $\mathcal{H} = \{H^{(1)} \times H^{(1)}, \dots, H^{(K)} \times H^{(K)}\}$.*

Proof: We are going to find a tiling of $A \times H$ first. Let us choose an integer m_1 which we will specify later and N such that $cN \geq M(m_1, N) + 1$. (Remember that for any m_1 fixed, $M(m_1, N)/N \rightarrow 0$ as $N \rightarrow 0$.) We can write $H(x_0, x_1, \dots, x_n)$ as $H(x_0, x_1, \dots, x_{n-N}) + H(0, x_{n-N+1}, \dots, x_n)$, that is, H consists of 2^{n-N} disjoint translates of a 2^N dimensional cube. Each translate has $c2^N$ elements from A in average. By the choice of N if a translate has at least $c2^N$ elements from A then it contains an m_1 dimensional cube which is in A . More elements, say, $c2^N + r$ elements provide more, at least $r + 1$ m_1 dimensional cubes which are in A . So, there are at least 2^{n-N} distinct m_1 dimensional cubes in A which are also subcubes of the translates. There are no more than $(m_1 + 2)^N/m_1!$ different subcubes in $H(0, x_{n-N+1}, \dots, x_n)$. (Note that $(m_1 + 2)^N/m_1!$ is a function of c and m_1 only.) Choose the most popular m_1 dimensional subcube H' , out of the $(m_1 + 2)^N/m_1!$ different types, of which

translate is most frequent in A . The set of translates of H' in A is denoted by $T = \{t_i + H' | 1 \leq i \leq s\}$, where $s \geq |H|m_1!/(2(m_1+2))^N$ and t_i -s are some elements from $H^-(x_0, x_1, \dots, x_{n-N})$. Our first tile (for $A \times H$) is $H' \times H'$. Remove the $|T|2^{n-N}$ translates of $H' \times H'$ from $A \times H$.

It was easy to find our first tile because the structure of $A \times H$ was simple. After removing the $|T|2^{n-N+2m_1}$ points, the remaining set is not that nice any more, however we can partition it into partition classes similar to our first set. We partition the remaining points into (not too many) partition classes of the form $A^* \times H^*$ where H^* is a cube and A^* is a large subset of H^* . After that in each partition class one can repeat the previous method of finding a new tile.

$A^* \times H^*$

As we did before, we write $H(x_0, x_1, \dots, x_n)$ as $H_1(0, x_{n-N+1}, \dots, x_n) + H_2(x_0, x_1, \dots, x_{n-N})$, that is H consists of 2^N disjoint translates of a 2^{n-N} dimensional cube. We can partition $H \times H = H(x_0, x_1, \dots, x_n) \times H(x_0, x_1, \dots, x_n)$ into 2^{2N} classes as

$$H \times H = \bigcup_{(v_1, v_2) \in H_1 \times H_1} (v_1, v_2) + H_2 \times H_2.$$

With this way of partitioning, in each partition class the pointsets have the form of $(A \cap (v_2 + H_2)) \times (v_1 + H_2)$. This is similar to the original $A \times H$ case with two differences; the left side of the product is not necessarily subset of the right side and it might be much smaller than the right side. The first is not a real difference since $A \cap (v_2 + H_2) + v_1 - v_2 \subset v_1 + H_2$ which is sufficient for us. In the second case, if $|A \cap (v_2 + H_2)| \leq \varepsilon/4|H_2|$ then we simply ignore the partition class, we do nothing with it. If $|A \cap (v_2 + H_2)| > \varepsilon/4|H_2|$ then we can find a good tile in it in the same way as we did earlier, now with a new $c_1 > \varepsilon/4$, with a smaller, 2^{n-N} dimensional, cube, and with the same m_1 . In the next iteration we repeat the algorithm for any partition class with $c_2 > \varepsilon/4$ density, with an even smaller dimensional cube, and with m_1 . In each iteration we find a tiling of at least the $\Theta(\varepsilon, m_1)$ fraction of the remaining points. To see that, let us follow another iteration on the Cartesian product $(A \cap (v_2 + H_2)) \times (v_1 + H_2)$ where $|A \cap (v_2 + H_2)| = c_1|H_2| > \varepsilon/4|H_2|$. Let $A^* \times H^* = (A \cap (v_2 + H_2)) \times (v_1 + H_2)$. Find N_1 such that $c_1 N_1 \geq M(m_1, N_1) + 1$. The value of N_1 depends on m_1 and $c_1 > \varepsilon/4$ only, therefore it is bounded by a uniform constant from above during the iterations. Write H^* as $H(x'_0, x_1, \dots, x_{n-N-N_1}) + H(0, x_{n-N-N_1+1}, \dots, x_{n-N})$. We selected N_1 that there are at least $2^{n-N-N_1} = |H_2|/2^{N_1}$ translates of m_1 dimensional cubes in A^* and at least $m_1!|H_2|/(2(m_1+2))^{N_1}$ of them are the same type. Each has cube has 2^{2m_1} points, so we find a tiling covering at least $|H_2|^2 m_1! 2^{2m_1} / (2(m_1+2))^{N_1}$ points. In each iteration we remove at least the $m_1! 2^{2m_1} / (2(m_1+2))^{N_1}$ -fraction of the points from any partition which was more than $\varepsilon/4$ -dense. After some L iterations, where L is a constant depending on ε and m_1 only, we find a tiling of $(A \times H) \setminus E$ by m_1 dimensional cubes where $|E| \leq \varepsilon/2|A||H|$.

In the next step we intersect every tile with $H \times B$. The total number of points of intersections is at least $(1 - \varepsilon/2)|A||B|$. The tiles are translates of $H' \times H'$ where H' is an m_1 dimensional cube. If $H \times B$ is at least $\varepsilon/4$ -dense in

$H \times B$

a tile $H' \times H' + (v_1, v_2)$, then find a $\varepsilon/4$ -complete tiling of it by m -dimensional tiles. It is always possible if we choose a large enough m_1 at the beginning of the proof. With n replaced by m_1 we repeat the partition algorithm we had for $A \times H$, now with $(H \times B) \cap (H' \times H' + (v_1, v_2))$. It is not important for the proof of Theorem 2.2, however it is worth mentioning that during the iterations we used a constant, $K = K(\varepsilon, m)$, types of tiles. ■

It remains to show that if $S \subset H \times H$ is corner-free then there is a large Cartesian product $A \times B$ where S has density larger than \bar{c} . Actually, what we are going to show is a somewhat weaker statement; there is a product of a cube $H' \times H' \subset H \times H$ where a large Cartesian product $A \times B$ is larger than \bar{c} .

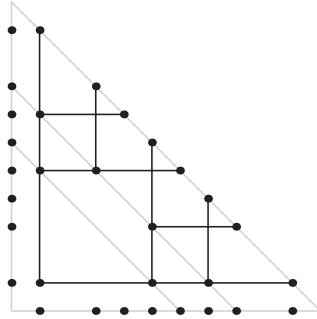
Let $H(x_0, x_1, \dots, x_n)$ be a cube and let $S \subset H \times H$ be a corner-free, c -dense subset, that is, $|S| = c|H|^2$.

diagonal

The *diagonal* of a product $H(x_0, x_1, \dots, x_n) \times H'(x'_0, x_1, \dots, x_n)$ is the set of points

$$Diag(H \times H') = \left\{ x_0 \vec{i} + x'_0 \vec{j} + \sum_{k=1}^n \nu_k x_k | \nu_k \in \{\vec{i}, \vec{j}\} \right\},$$

where \vec{i} and \vec{j} are the axis parallel unit vectors. The points of the diagonal are on the line which connects two opposite corners of the square defined by the Cartesian products of the cubes. (It is the top line with slope -1 in the picture below)



In what follows we will find a product, $H' \times H'' \subset H \times H$, where S is almost c -dense and $|S \cap Diag(H' \times H'')| \geq \frac{c}{30} |H''|^2$. Then, either we will find a corner in S with two vertices in $S \cap Diag(H' \times H'')$, or there is a large ($\geq \frac{c}{30} |H''|$) and almost empty Cartesian product, induced by $S \cap Diag(H' \times H'')$, which forces a large and dense Cartesian product somewhere in $H' \times H''$. In the first case we are done and in the second we can apply Lemma 2.5 to find a product of a cube in $H' \times H''$ in which S has density larger than \bar{c} .

In order to find a Cartesian product described above we apply some basic tools from probability; a Chernoff-type bound and the linearity of expectations. (an excellent resource to learn about probabilistic methods in combinatorics is the book of Alon and Spencer [1].)

There is a geometric description of the points of $H \times H$ by the axis parallel unit vectors:

$$H \times H = \left\{ x_0(\vec{i} + \vec{j}) + \sum_{k=1}^n \nu_k x_k | \nu_k \in \{0, \vec{i}, \vec{j}, \vec{i} + \vec{j}\} \right\}.$$

Set S has $c4^n$ elements, each can be represented by a point of $[4]^n$ as follows; if ν_k is $0, \vec{i}, \vec{j}$, or $\vec{i} + \vec{j}$ then the k -th coordinate of the corresponding point is 1, 2, 3, or 4 (resp.) For our calculations we need to observe that most of the points of S have about half of its coordinates 2 or 3 (as expected.) A Chernoff type bound gives that the number of points in $[4]^n$ which have less than $n/2 - \Delta\sqrt{n}$ positions with 2-s and 3-s is less than $4^n e^{-\frac{\Delta^2}{2}}$. For any $\varepsilon_1 > 0$ we can choose Δ large enough that at least $(1 - \varepsilon_1)|S|$ points have at least $n/2 - \Delta\sqrt{n}$ positions with 2-s and 3-s. Let us choose $\frac{\sqrt{n}}{2\Delta}$ elements of $\{x_1, \dots, x_n\}$ at random. The random set is denoted by Λ and the cube generated by Λ is H' . Then $H = H'' + H'$ where H'' denotes the cube generated by $\{x_0, x_1, \dots, x_n\} \setminus \Lambda$. The product $H \times H$ can be written as

Chernoff

ε_1

$$H \times H = \bigcup_{v \in H'' \times H''} v + (H' \times H').$$

It gives a complete tiling of $H \times H$ by $H' \times H'$. An element of S is on the diagonal of a cube $v + (H' \times H')$ for some $v \in H'' \times H''$ iff Λ is a subset of its set of 2-s and 3-s. The probability of that (at least for the $(1 - \varepsilon_1)|S|$ typical elements) is

$$Prob \geq \left(\frac{n/2 - \Delta\sqrt{n}}{n} \right)^{|\Lambda|} = \left(\frac{1}{2} - \frac{\Delta}{\sqrt{n}} \right)^{|\Lambda|} \approx e^{-1} 2^{-\frac{\sqrt{n}}{2\Delta}}$$

The product $H \times H$ is partitioned into $4^{n-|\Lambda|}$ classes each consists of a translate of $H' \times H'$ where H' is a $|\Lambda|$ dimensional cube. The cardinality of diagonal points all together is $4^{n-|\Lambda|} 2^{|\Lambda|} = 4^n 2^{-|\Lambda|}$. From the previous calculations it follows that if $\varepsilon_1 < 1 - e/3$ then at least

$$\frac{c(1 - \varepsilon_1)4^n}{e \cdot 2^{|\Lambda|}} > \frac{c4^n}{3 \cdot 2^{|\Lambda|}} \quad (1)$$

diagonal points are from S . Then, there are more than $\frac{3}{10}4^{n-|\Lambda|}$ tiles, $v + (H' \times H')$, where the diagonal contains at least $\frac{c}{30}2^{|\Lambda|}$ points from S . Let us suppose that in all these tiles S has density less than $c - \varepsilon_2$, for some $\varepsilon_2 > 0$ which we will specify in the next step. The $\frac{3}{10}4^{n-|\Lambda|}$ low density tiles push the average density for the remaining tiles to be at least $c + \frac{3\varepsilon_2}{7}$. Choose ε_2 such that

ε_2

$$c + \frac{3\varepsilon_2}{7} \geq \bar{c}. \quad (2)$$

Either there is a tile where S is very dense in which case we are done as there should be a corner in S , or we can suppose that there is a tile, $v + (H' \times H')$, where

the diagonal contains at least $\frac{c}{30} 2^{|\Lambda|}$ points from S , and the density of S is at least $c - \varepsilon_2$. The points of S on the diagonal $Diag(v + (H' \times H'))$ determine a Cartesian product which is the product of the projection of $Diag(v + (H' \times H')) \cap S$ onto the x -axis, denoted by D_x , and the projection onto the y -axis, denoted by D_y . (It is not obvious from the notation however it should be clear that D_x and D_y depend on v .) If S is corner-free then the product $D_x \times D_y$ has points from S only on $Diag(v + (H' \times H'))$. Then $D_x \times D_y$ has negligible density and therefore $\overline{D_x} \times D_y$, $D_x \times \overline{D_y}$, or $\overline{D_x} \times \overline{D_y}$ has larger density. ($\overline{D_x}$ is the complement of D_x in $v_x + H'$.) Simple calculation shows that this increased density is at least

$$\frac{c \cdot 30^2}{30^2 - (c - \varepsilon_2)^2} > c + \frac{c^3}{60^2} \quad (3)$$

if $\varepsilon_2 < c/2$. The proof is complete. We can apply Lemma 2.5 to find a large and dense product of a cube in one of the products $\overline{D_x} \times D_y$, $D_x \times \overline{D_y}$, or $\overline{D_x} \times \overline{D_y}$. For example, let us suppose that $|D_x \times \overline{D_y} \cap S| \geq (c + \frac{c^3}{60^2}) |D_x| |\overline{D_y}|$ and that $\frac{c}{30} |H'| \leq |D_x| \leq |\overline{D_y}|$. We can also suppose that $c + \frac{c^3}{2 \cdot 60^2} > \bar{c}$. Using the notation of Lemma 2.5 we set $\delta = c + \frac{c^3}{60^2}$, $\varepsilon < \frac{c^3}{2 \cdot 60^2}$, $\alpha = \frac{c}{30}$, $A = D_x$, and $B = \overline{D_y}$. With these parameters the lemma guarantees the existence of a product of a large cube were S has density more than \bar{c} which is not possible if S was corner-free.

3 Density Hales-Jewett for $k = 3$

The proof of our previous result contains some of the key elements of the proof of the density version of the Hales-Jewett theorem for $k = 3$. In this section we are going to sketch the proof. We state the theorem (DHJ(k=3)) using Cartesian products. As before, we are given a Cartesian product of a cube $H\{x_0, x_1, \dots, x_n\}$, but this time we consider a subset of the points only:

$$\widetilde{H \times H} = \left\{ x_0(\vec{i} + \vec{j}) + \sum_{k=1}^n \nu_k x_k \mid \nu_k \in \{0, \vec{i}, \vec{j}\} \right\} \subset H \times H.$$

There is a 1 – 1 correspondence between $\widetilde{H \times H}$ and $[3]^n$. DHJ(k=3) is equivalent to the following

Theorem 3.1 *If the dimension of a cube H is sufficiently large depending on $\delta > 0$, then any δ -dense subset of $\widetilde{H \times H}$ must contain a “corner”, three points having coordinates (x, y) , $(x + r, y)$, $(x, y + r)$ with $r \neq 0$. (Note that in this case $r \neq 0$ implies $r > 0$.)*

Before we describe the proof, let us review the basic steps of the proof of Theorem 2.2. We found a partition of $H \times H$ into disjoint translates of a “tile” $H' \times H'$ where a typical translate has many point on the diagonal from S . If all partition classes with many points on the diagonal (from S) had small

density then there would be many classes with small density, so there would be a partition class having large density and we would be done. Therefore, we can suppose that there is a partition, say $H' \times H'$, with expected density and with many points on the diagonal, i.e. $\text{diag}(H' \times H') \cap S$ is large. Since S is corner-free, the points of S on the diagonal induce a large and almost empty Cartesian product which increases the density on a large Cartesian product somewhere in $H' \times H'$ (in one of the products $\overline{D_x} \times D_y$, $D_x \times \overline{D_y}$, or $\overline{D_x} \times \overline{D_y}$.) Then, by Lemma 2.6, there is a product of a cube with large density which completes the proof.

The next example shows that it is not enough to repeat the same arguments as above in order to prove Theorem 3.1.

Example 3.2 *Select half of the points of $\text{diag}(H \times H) = \text{diag}(\widetilde{H \times H})$ as follows*

$$D := \left\{ x_0 \vec{i} + x'_0 \vec{j} + x_1 \vec{i} + \sum_{k=2}^n \nu_k x_k | \nu_k \in \{\vec{i}, \vec{j}\} \right\}.$$

With such selection, no pair in D gives a corner with any point of $\widetilde{H \times H}$, i.e. $(D_x \times D_y) \cap (\widetilde{H' \times H'}) = D$.

The above example suggests that just having many points on a diagonal is not of much help. We need to show the stronger statement that for any dense set S , a random partitioning results in many pairs on the diagonals such that the third point of the corner determined by such a pair is in $H' \times H'$. Since this is the main difficulty of the whole proof, we have to be more precise here:

As before, we work with a random set $\Lambda \subset \{x_1, \dots, x_n\}$. The cube generated by Λ is H' . Then $H = \widetilde{H'' + H'}$ where H'' denotes the cube generated by $\{x_0, x_1, \dots, x_n\} \setminus \Lambda$. $\widetilde{H \times H}$ can be written as

$$\widetilde{H \times H} = \bigcup_{v \in \widetilde{H'' \times H''}} v + (\widetilde{H' \times H'}).$$

It gives a complete tiling of $\widetilde{H \times H}$ by $\widetilde{H' \times H'}$.

For a density increment argument we need to show that most diagonals have many good pairs.

Lemma 3.3 *For any real number $c_1 > 0$ there is a $\delta = \delta(c_1) > 0$ such that if S is a c_1 -dense subset of $\widetilde{H \times H}$ then the set of vectors*

$$V = \left\{ v \in \widetilde{H'' \times H''} : \left| (D_x \times D_y) \cap (v + \widetilde{H' \times H'}) \right| \geq \delta \left| \widetilde{H' \times H'} \right| \right\}$$

has cardinality at least $\delta \left| \widetilde{H'' \times H''} \right|$.

The second ingredient which is needed for the proof is a variant of Lemma 2.5.

Lemma 3.4 *For any reals $\varepsilon > 0$, $\alpha > 0$ and integer $m > 2$ there is a threshold n_0 that the following holds; if A and B are subsets of an n dimensional cube H , $n \geq n_0$, $|A \times B \cap \widetilde{H \times H}| \geq \alpha |\widetilde{H \times H}|$, and S is a δ -dense subset of $A \times B \cap \widetilde{H \times H}$, then there is an m dimensional cube H' and a vector v such that S is at least $(\delta - \varepsilon)$ -dense on $v + \widetilde{H' \times H'}$.*

The proof of Lemma 3.4 goes along the lines of the proof of Lemma 2.5. Whenever we used $\widetilde{H \times H}$, $\widetilde{H' \times H'}$, or $\widetilde{H'' \times H''}$, in the new proof we use $\widetilde{H \times H}$, $\widetilde{H' \times H'}$, and $\widetilde{H'' \times H''}$, resp. We omit the proof.

Armed with Lemma 3.3 and Lemma 3.4 we could complete the proof of Theorem 3.1 exactly as we did in the Cartesian product case in the previous section (Theorem 2.2). Unfortunately – while Lemma 3.4 is easy to prove – I don't see a similarly simple proof for Lemma 3.3. Let us look at the rough numbers at least, see why the statement should be true.

Like in the Cartesian product case, a point of S is a diagonal point iff Λ is a subset of its set of 2-s and 3-s. (The points are represented as n -length strings of 1-s, 2-s, and 3-s.) Two diagonal points are points of the same diagonal iff they are identical outside of Λ . Two points of the same diagonal are “good” (the third corner point is in $\widetilde{H \times H}$) iff the set of 2-s of one point is a subset of the 2-s of the other. Two points of S form a *potentially good pair* if they have the the same set of 1-s and the set of 2-s of one contains the set of 2-s in the other one. What is the probability that a potentially good pair is on a diagonal after choosing Λ ? That means that both points have only 2-s and 3-s in Λ and all positions where they are different are in Λ . Suppose that the first has a 1-s, b 2-s, and c 3-s, and the second has a 1-s, $b - d$ 2-s, and $c + d$ 3-s. The size of Λ is denoted by N . Then the probability that both points are on the same diagonal is

$$\left(\frac{b+c}{n}\right)^N \binom{b+c-d}{N-d} \binom{b+c}{N}^{-1}.$$

If the number of potentially good pairs where the first has b 2-s, and c 3-s, and the second has $b - d$ 2-s, and $c + d$ 3-s is denoted by $\xi_{b,c,d}^{(n)}$ then Lemma 3.3 can be stated as follows; *for every $c_1 > 0$ there is a $\delta' > 0$ such that*

$$\sum_{b,c,d} \xi_{b,c,d}^{(n)} \left(\frac{b+c}{n}\right)^N \binom{b+c-d}{N-d} \binom{b+c}{N}^{-1} \geq \delta' 3^n. \quad (4)$$

We did not specify the range of the sum, however Chernoff-type bounds suggest that it will be enough if the summation goes for $n/3 - \Delta\sqrt{n} \leq b, c \leq n/3 + \Delta\sqrt{n}$, and $N/3 - \sqrt{N} \leq d \leq N/3 + \sqrt{N}$, for some large constant Δ , and the value of N will be selected later.

It remains to give a bound on $\xi_{b,c,d}^{(n)}$. Let s be an element of S with b 2-s and c 3-s. The number of permutations in which the digits of s are in increasing order (like 1111223333) is $b!c!(n-b-c)! \approx ((n/3)!)^3$. Two elements form a potentially

good pair iff the number of 1-s is the same in both elements and there is a permutation which moves the characters into increasing order in both. The number of potentially good pairs (counting with multiplicity) is at least

$$n!4\Delta\sqrt{n}\binom{\left(\frac{n}{3}\right)^3 c_1 3^n}{2} \approx c_2 n!n^{3/2},$$

where c_2 depends on c_1 and Δ only. A positive fraction of these pairs have at least $\gamma\sqrt{n}$ different coordinates where one is 2 and the other one is 3 (i.e. when $d \geq \gamma\sqrt{n}$ in $\xi_{b,c,d}^{(n)}$.) By choosing a small enough γ we can suppose that at least $\frac{c_2}{2}n!n^{3/2}$ potentially good pairs have this property. Any element of $\xi_{b,c,d}^{(n)}$ was counted with multiplicity $(n-b-c)!(b-d)!d!c!$. After some simplifications we get the inequality

$$\sum_{d=\gamma\sqrt{n}}^{2\Delta\sqrt{n}} \sum_{\{b,c\}=n/3-\Delta\sqrt{n}}^{n/3+\Delta\sqrt{n}} \xi_{b,c,d}^{(n)} \binom{c}{d}^{-1} \geq \frac{c_2}{2} 3^n \sqrt{n}. \quad (5)$$

There is a number M such that $\gamma\sqrt{n} \leq M \leq 2\Delta\sqrt{n}$ and

$$\sum_{d=M-\sqrt{M}}^{M+\sqrt{M}} \sum_{\{b,c\}=n/3-\Delta\sqrt{n}}^{n/3+\Delta\sqrt{n}} \xi_{b,c,d}^{(n)} \binom{c}{d}^{-1} M^{-1/2} \geq c_3 3^n. \quad (6)$$

Inequality (6) can be compared with (4) if we choose $N = 3M$. In the range of the summation the coefficient of $\xi_{b,c,d}^{(n)}$ in (6) is a constant multiplier of the coefficient in (4) therefore the inequality in (4) holds and so does Lemma 3.3.

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