Discrete Math

subsets

Problem 1. *Find the number of subsets of* $\{1, \dots, n\}$ *having an even number of elements.*

Solution.

$$
\begin{aligned}
&|\{A \subset \{1, \dots, n\} : |A| \text{ is even}\}| \\
=&\sum_{k \text{ even}} \binom{n}{k} \\
=&\sum_{k=0}^{n} \binom{n}{k} \left(\frac{1 + (-1)^k}{2}\right) \\
=&\frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} + \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} \\
=&\frac{1}{2} \times 2^n + \frac{1}{2}(-1 + 1)^n \\
=&2^{n-1}
\end{aligned}
$$

□

Problem 2. *Find the number of subsets of* $\{1,\ldots,n\}$ *whose number of elements is a multiple of 3.*

Solution. Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Then $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.

$$
\begin{aligned}
&|\{A \subset \{1,\ldots,n\} : |A| \equiv 0 \pmod 3\}| \\
=&\sum_{k\equiv 0 \pmod 3} \binom{n}{k} \\
=&\sum_{k=0}^{n} \binom{n}{k} \times \frac{1}{3}\left(1 + \omega^k + \omega^{2k}\right) \\
=&\frac{1}{3}\sum_{k=0}^{n}\binom{n}{k} + \frac{1}{3}\sum_{k=0}^{n}\binom{n}{k}\omega^k 1^{n-k} + \frac{1}{3}\sum_{k=0}^{n}\binom{n}{k}(\omega^2)^k 1^{n-k} \\
=&\frac{2^n}{3} + \frac{1}{3}(1+\omega)^n + \frac{1}{3}(1+\omega^2)^n \\
=&\frac{2^n}{3} + \frac{1}{3}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^n + \frac{1}{3}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^n \\
=&\frac{2^n}{3} + \frac{1}{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^n + \frac{1}{3}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^n \\
=&\frac{2^n}{3} + \frac{1}{3}\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + \frac{1}{3}\left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right) \\
=&\frac{2^n}{3} + \frac{2}{3}\cos\frac{n\pi}{3} \\
=&\begin{cases}
\frac{2^n+1}{3} & \text{when } n \equiv \pm 1 \pmod 6 \\
\frac{2^n-1}{3} & \text{when } n \equiv \pm 2 \pmod 6 \\
\frac{2^n-2}{3} & \text{when } n \equiv 3 \pmod 6 \\
\frac{2^n+2}{3} & \text{when } n \equiv 0 \pmod 6
\end{cases}
\end{aligned}
$$

□