MATH 503 Assignment 1

Problem 1. We wish to pick out subsets of an n element set whose cardinality is divisible by 3. Recalling the simple formula which counts all subsets of an n elements set:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

it follows immediately that the desired quantity is:

$$\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{3k} \approx \frac{2^n}{3}$$

It remains then to find an exact closed formula. Using a standard technique, we first note that if ω is the third root of unity, namely $e^{i2\pi/3}$, then $\omega^3 = 1$ and $1+\omega+\omega^2 = 0$. Thus $1+\omega^k+\omega^{2k} = 3$ if $3|k$ and 0 otherwise. Finally, recall that $(1+\omega^\alpha)^n = \sum_{k=0}^{n} \binom{n}{k}\omega^{\alpha k}$. Thus we have:

$$\begin{aligned}
\sum_{k=0}^{\lfloor \frac{n}{3} \rfloor} \binom{n}{3k} &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{3}(1+\omega^k+\omega^{2k}) \\
&= \frac{1}{3}(2^n + (1+\omega)^n + (1+\omega^2)^n) \\
&= \frac{1}{3}(2^n + e^{in\pi/3} + e^{i5n\pi/3}) \\
&= \frac{2^n}{3} + \frac{1}{3}(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} + \cos\frac{5n\pi}{3} + i\sin\frac{5n\pi}{3}) \\
&= \frac{2^n}{3} + \frac{1}{3}(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} + \cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}) \\
&= \frac{2^n}{3} + \frac{2}{3}(\cos\frac{n\pi}{3}) \\
&= \frac{2^n + 2(\cos\frac{n\pi}{3})}{3}
\end{aligned}$$

Finally recall that:

$\cos\frac{n\pi}{3} = \frac{1}{2}$ when $n \equiv \pm1(\mod 6) \implies \frac{2^n+1}{3}$
$\cos\frac{n\pi}{3} = -\frac{1}{2}$ when $n \equiv \pm2(\mod 6) \implies \frac{2^n-1}{3}$
$\cos\frac{n\pi}{3} = -1$ when $n \equiv \pm3(\mod 6) \implies \frac{2^n-2}{3}$
$\cos\frac{n\pi}{3} = 1$ when $n \equiv \pm0(\mod 6) \implies \frac{2^n+2}{3}$

This shows the desired result.

Problem 2. There are many ways to attack this problem. Firstly, note that a connected graph with exactly one cycle has on n vertices has exactly n edges. This follows from the fact that a tree is a connected graph with exactly $n-1$ edges. Thus consider the graph with the edges of the unique cycle removed. This graph is then a rooted forest with each root acting as one of the vertices of the cycle. Further note that in a simple graph, each cycle must have length at least three. A naive approach to this problem is to try to start with labled trees and simply count all the ways to add an edge that results in a cyle. However there are numerous symmetric cases that arise and are difficult to keep track of. Instead we will make use of the fact that the number of rooted forests with $r \leq n$ specified roots is rn^{n-r-1}. The proof of this is analogous to the proof of Cayley's formula using Pruefer codes. Note also that when $r = 1$ this gives the number of rooted labled trees with a single specified root, as well as the number of labled trees.

Now consider the unique cycle of the graph. This cycle has length ranging from 3 to n (we ommit the trivial cases when n is less than 3). Removing the edges of this cycle we then have $k \in \{3, \ldots, n\}$ vertices which we specify as the k roots of our rooted forest. Now the number of rooted forests with k specified roots is kn^{n-k-1}. Thus it suffices to count the number of different ways these roots can compose the cycle. In particular, there are n choices for the first root, $n-1$ choices for the second root, and so on, finally giving $n-k+1$ choices for the last root. Using the notation from class, there are $(n)_k$ ways in which we can form the cycle. However these cycles are not unique as we can rotate the cycle through k different starting positions. Furthermore we pick the vertices in both a clockwise and counterclockwise fashion. Thus the number of unique cycles on k vertices is:

$$\frac{(n)_k}{2k}$$

Putting this together with the number of rooted trees on k specified roots we obtain given k that there are :

$$\frac{(n)_k}{2k} kn^{n-k-1}$$

connected labled graphs with a single cycle of length k. Finally summing over all possible k gives:

$$\sum_{k=3}^{n} \frac{(n)_k}{2k} kn^{n-k-1}$$

a bit of rearranging gives the slightly nicer form:

$$n^{n-1} \sum_{k=3}^{n} \frac{(n)_k}{2n^k}$$

Problem 3. Recall that stirling numbers of the second kind, denoted $\left\{ n \atop k \right\}$ indicate the number of ways a set of n elements can be partioned into k non-empty sets. Following the hint in Lovasz, we note that any partition of n elements into $n-k$ classes has at least $n-2k$ singletons. Now consider the sets with more than 1 element. Lets call these s and the union of all such sets S. Then $|S| \leq 2k$. Denote $|S|$ as l. Now it suffices to consider the number of ways in which on can parition these l elements into $l-k$ sets of cardinality greater than or equal to 2. Lets call this quanity s_k^l which is an integer. It then follows that given l there are $\binom{n}{l}$ ways to pick l and s_k^l ways to parition these l elements. Finally recall that l can range from 0 to $2k$. Putting this all together we obtain:

$$\left\{ n \atop n-k \right\} = \sum_{l=0}^{2k} s_k^l \binom{n}{l}$$

which is a polynomial in n given k as required.

Problem 4.

1. There are may ways to interpret this. One way is as follows: For the left hand side, you are choosing k elements from a set of m elements. Then taking a set of n elements, add the chosen k elements and pick m elements from this. This can be seen as an ordered pair (x_k, x_m). To see the right hand side, consider instead picking the elements that are in both the m set and the $n+k$ set. Some portion of these say l will be in both. Thus from the set n element set we pick l. These l elements are now fixed, but we must pick the rest of the m set elements, namely $m-l$ such elements. Thus there are $\binom{n}{l}\binom{m}{m-l}$ ways to do this. By the symmetry of the binomial operator this implies there are $\binom{n}{l}\binom{m}{l}$ ways to do this. Finally we need to pick x_k elements. There are 2^l ways to do this. The result follows.

2. Again there are many ways to attack this. Perhaps the easiest is to notice that the left hand side double counts (due again to the symmetry of the choose operator) the number of rooted forests with exactly two trees on n vertices. The n^{n-2} on the right hand side counts the number of labled trees on n vertices. The two on the right hand side accounts for the double counting of the left hand side, while $n-1$ is the number of edges that a labed trees on n vertices has. Thus given a labled tree on n vertices one can pick any of the $n-1$ edges to break, creating two rooted trees with the vertices incident to the edge acting as the roots. Thus the two sides count the same items and the result follows as required.