Question 1. What is the probability that a random permutation of \( n \) objects have a fixpoint? (Every permutation is selected with the same, \( \frac{1}{n!} \), probability.) What is the expected number of fixpoints?

Question 2. Prove that \( \binom{n}{k} \) and \( \binom{2n}{2k} \) have the same parity.

Question 3. Use Szemerédi’s regularity lemma to prove that for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that, for any graph \( G_n \) on \( n \) vertices with at most \( \delta n^3 \) triangles, it may be made triangle-free by removing at most \( \varepsilon n^2 \) edges.

Question 4. Use the eigenvalues of the adjacency matrix to give an exact formula for the number of pentagons \( (C_5 \text{-s}) \) in a graph.

Question 5. Let us suppose that the second largest eigenvalue of a \( d \)-regular bipartite graph is \( 2\sqrt{d} \). The two vertex sets have size \( n - n \). Give upper and lower bounds on the number of quadrilaterals \( (C_4 \text{-s}) \) in the graph.

Question 6.* Show that there is a constant, \( c > 0 \), such that the following holds; let \( P \) and \( Q \) be a set of points and a set of axis parallel rectangles in the plane, respectively. Let us suppose that every rectangle contains at least one point. Then for any natural number \( n \), either

- there are \( n \) rectangles in \( Q \) such that no point of \( P \) belongs to more than one of them, or
- one can choose at most \( n^c \) points of \( P \) so that every element of \( Q \) contains at least one of them.

Due date: Dec. 11, 11:00 pm. Please send your solutions by email.