Question 1. Answer the following question using the Lovász Local Lemma. Given a positive integer \( k \in \mathbb{Z} \). How many colours do we need to colour the integer grid, \( \mathbb{Z} \times \mathbb{Z} \), such that every square of side length \( r = \sqrt{k} \) has four different different colours in its vertices. The squares might not be axis parallel. (The squares are spanned by \( \mathbb{Z} \times \mathbb{Z} \).)

Question 2. What is the probability that a random sequence of length \( n - 2 \) is the Prüfer code of a path? Every entry of the sequence is chosen independently at random with equal \((1/n)\) probability from the set \( \{1, \ldots, n-1, n\} \).

Question 3. In class we sketched the following bound on the diagonal Ramsey number \( R(k,k) \): ... If we set \( n \) such that with the random colouring of \( K_n \) (colour the edges with red or blue independently at random with probability \( 1/2 \)) the expected number of monochromatic \( k \)-cliques is at most \( n/k \) then we have \( R(k,k) \geq n - n/k \). What is the bound we get from this? Use Stirling’s formula to perform the exact calculations.

Question 4. A planar curve \( \gamma \) is \( x \)-monotone if any vertical line either does not intersect \( \gamma \), or it intersects \( \gamma \) in a single point. It is \( y \)-monotone if any horizontal line either does not intersect \( \gamma \), or it intersects \( \gamma \) in a single point. If both conditions hold then \( \gamma \) is \( xy \)-monotone. Give a lower bound on the function \( g(n) \) which is defined as follows; it doesn’t matter how we select \( n \) points with distinct \( x \) and \( y \) coordinates, at least \( g(n) \) of them lie on an \( xy \)-monotone continuous planar curve \( \gamma \).

Due date: Oct. 8, in class.