MATH 503 HW 2 SOLUTIONS

Question 1. Let us colour all subsets of an n-element set by $n-1$ colours.

(1) Show that there are two sets, A and B having the same colour so that one is a subset of the other, $A \subset B$.

A: There is a chain of length n where two elements are having the same colour since the number of colours is $n-1$.

(2) What is the expected number of such monochromatic $A \subset B$ pairs if we colour the sets independently at random using the $n-1$ colours with equal $(1/(n-1))$ probability for each set?

A: Let us denote the number of $A \subset B$ pairs by M. (Not counting $A = B$) For every pair the probability that the pairs have the same colour is $1/(n-1)$. By the linearity of expectation the answer to the question is $M/(n-1)$. To find the value of M let us colour the elements using colours red, blue, and white. For every $A \subset B$ there is a unique colouring where A is red and B is the union of the red and blue elements. Therefore $M = 3^n - 2^n$, and the expected number of monochromatic pairs is $(3^n - 2^n)/(n-1)$.

(3) Answer the previous two questions if we use $k \leq n$ colours instead of $n-1$.

A: If $k = n$ then there is a colouring without any $A \subset B$ monochromatic pair; colour different cardinality sets with different colours. If $0 < k < n$ then the same calculations work writing k instead of $n-1$ for the number of colours.

Question 2. What is the probability that a random sequence of length $n-2$ is the Prüfer code of a star? Every entry of the sequence is chosen independently at random with equal $(1/n)$ probability from the set $\{1, \ldots, n-1, n\}$.

A: The Prüfer code of a star is a sequence of the form $\overbrace{a, a, \ldots, a}^{\text{n-2}}$ where $a \in [n]$. Prob=n^{-n+3} $(n \geq 3)$ and Prob=1 if $n = 2$.