MATH 309 PRACTICE QUESTIONS FOR FINAL - SOLUTIONS

Solutions The convex hull of a set A is the intersection of all the convex sets containing A.

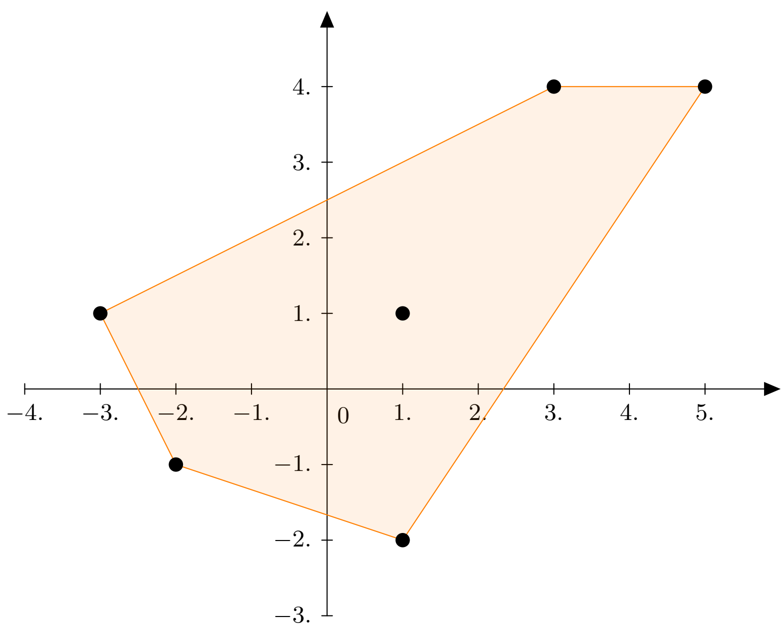
A set H is a convex set if for any two points $x, y \in H$, the line segment joining x and y lies in H.

Solutions It is never true, unless they share the same centre.

WLOG, assume the centres of the balls are $(0,0,0)$ and $(a,0,0)$. Then for $0 < \varepsilon < 1$, the two points $P = (0,0,1-\varepsilon)$ and $Q = (a,0,1-\varepsilon)$ are both contained in the union of the two balls. Now, it is easy to see that the mid-point of PQ does not lie in any of the balls, provided that $\varepsilon > 0$ is small enough.

Solutions The chromatic number $\chi(G_n)$ is the smallest number of colours needed to colour the vertex set of the graph G_n so that the two endpoints of every edge always receive different colours.

Solutions The convex hull is the closed set as shown.



Solutions

- Suppose every vertex has degree at least 4. Then,
$$2e = \sum_v \deg(v) \geq 4n.$$
Also, by double counting the sum of edges in facets (note $f_3 = 0$), we have
$$2e = \sum_{i=4}^{n} i f_i \geq 4f.$$
Together, we have $4e \geq 4n + 4f$, or
$$e \geq n + f.$$
By Euler's formula, $n - e + f = 2$, and so
$$n + f - 2 = e \geq n + f,$$
which implies $-2 \geq 0$, contradiction.
- We use induction on n. The base case is trivial.

  Now, suppose that every simple planar n-vertex graph G_n which has only facets with even number of edges on the boundary is 4-colourable. Let $G = G_{n+1}$ be a graph on $n+1$ vertices satisfying the assumption. By the previous part, there is a vertex v which has degree $d \leq 3$.

  We will show that every facet of the graph $G' = G - v$ has an even number of edges on the boundary. Indeed, the facets containing the vertex v in G become a new facet in G' (the number of edges of this new facet in G' is the total number of edges of the facets containing v in G minus $2d$, which is even), other facets in G remains the same in G' (each of them has an even number of edges on the boundary).

  By induction hypothesis, the graph G' is 4-colourable. Since the degree of v in G' is at most 3, there is at least one available colour for the vertex v. Hence, G' is also 4-colourable.

Solutions We note that the statement would not hold if we allow the squares to (partially) share an edge, but if we do not allow that, then it is true:

Colour a region red if it is contained in an odd number of squares, and blue otherwise, then no neighbouring regions receive the same colour.

Solutions The crossing number of a graph G is the smallest number of edge crossings with which the graph can be drawn.

Solutions Obtain a planar graph G'_n by removing at most $cr(G_n)$ edges. So G'_n has at least $e(G_n) - cr(G_n)$ edges. Note that $2e(G'_n) \geq 3f(G'_n)$. By Euler's formula, we have
$$2 = n - e(G'_n) + f(G'_n) \leq n - \frac{e(G'_n)}{3} \leq n - \frac{e(G_n) - cr(G_n)}{3}.$$
or $cr(G_n) \geq e(G_n) - 3v(G_n) + 6$.

Let H_p be a random subgraph of $G = G_n$ formed by retaining a vertex with probability p. An edge (u, v) in G is retained in H_p if both u and v are in $V(H_p)$.

By the above, we have

$$cr(H_p) \geq e(H_p) - 3v(H_p) + 6.$$

From the linearity of expectation we know

$$\mathbb{E}(cr(H_p)) \geq \mathbb{E}(e(H_p)) - 3\mathbb{E}(v(H_p)) + 6.$$

Note that a crossing survives in H_p with a probability p^4. Hence, $\mathbb{E}(cr(H_p)) = p^4(cr(G))$. By definition we have $\mathbb{E}(e(H_p)) = 7np^2$ and $\mathbb{E}(v(H_p)) = pn$. Thus,

$$p^4(cr(G)) \geq 7np^2 - 3pn + 6,$$

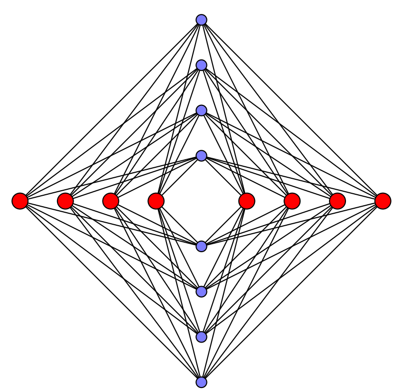
or

$$cr(G) \geq \frac{7np^2 - 3pn + 6}{p^4} > \frac{7p - 3}{p^3}n.$$

The right hand side is maximized when $p = 9/14$. This gives $cr(G_n) \geq 5.646n$.

Solutions

- Upper bound: Consider a drawing of $K_{n,n}$ with $cr(K_{n,n})$ crossings. Pick two vertices from each of the two vertex classes, there are at most two crossings. Hence, the total number of crossings is at most $2(\binom{n}{2})^2 \sim n^4/2$.
- Lower bound: Draw the vertices of one class on the x-axis and the other class on the y-axis, as shown in the figure below (when $n = 8$). This drawing of $K_{n,n}$ has about n^4 crossings.



Hence, the magnitude of $cr(K_{n,n})$ is n^4.

Solutions Given n points and m lines in the plane, the number of incidences is at most $c(n^{2/3}m^{2/3} + n + m)$, for some constant $c > 0$.

Proof. We may assume that every line contains at least three of the n points. (Because those lines containing at most two points contributes at most $2m$ incidences.)

Consider the graph G on these n points and the line segments as edges. (If a line contains k points, then it gives $k - 1$ line segments, and hence $k - 1$ edges in G.) We note that $cr(G) \leq m^2$.

Now, by the crossing number inequality, either $e(G) \leq 4n$, or

$$m^2 \geq cr(G) \geq \frac{c' e(G)^3}{n^2},$$

giving $e(G) \leq c''(n + n^{2/3}m^{2/3})$.

Finally, by the definition of the edge set of G, the number of incidences is $e(G) + m$, which is at most

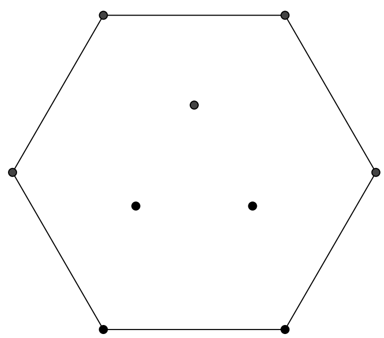
$$c(m + n + n^{2/3}m^{2/3}).$$

□

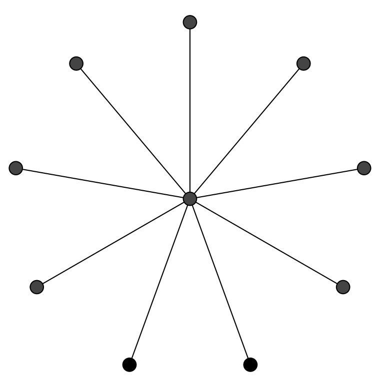
Solutions Using Szemerédi-Trotter for $\sqrt{n}$-rich lines, the maximum number of $\sqrt{n}$-rich lines is

$$c\frac{(n^2)^2}{(\sqrt{n})^3} = cn^{5/2}.$$

Solutions



Solutions



Solutions For $k, \ell \geq 2$, every set of $\binom{k+\ell-4}{k-2}$ points in general position contains a k-cup or an ℓ-cap.

Solutions

$$\begin{aligned}\binom{n-1}{k} + \binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!} \\ &= \frac{(n-1)!}{(k-1)!(n-k-1)!}\left(\frac{1}{k} + \frac{1}{n-k}\right) \\ &= \frac{(n-1)!}{(k-1)!(n-k-1)!}\frac{n}{k(n-k)} \\ &= \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k}\end{aligned}$$

Solutions

- $[2, 0], [0, 2]$.
- Yes. The density is $\pi/4$.

Solutions

- $[-2, -4, -5], [2, 2, 2], [-3, 3, -3]$.
- Yes. The volume of a fundamental parallelpiped is

$$\begin{vmatrix} 2 & 4 & 5 \\ -2 & -2 & -2 \\ 3 & -3 & 3 \end{vmatrix} = 36.$$

Hence, the density is $\frac{4\pi/3}{36} = \pi/27$.

Solutions Suppose we have k points in an $n \times n$ integer grid forming a convex polygon. We consider the graph G consisting of all of its translates in the $2n \times 2n$ grid. There are about n^2 such translates. We note that any two of these translates can produce at most two crossings. Hence

$$cr(G) \leq 2\binom{n^2}{2} \sim n^4.$$

Since an edge can be shared by at most two of such translates, there are at least $kn^2/2$ edges on $(2n)^2 = 4n^2$ vertices. By the crossing number inequality we have

$$cr(G) \geq c\frac{(kn^2/2)^3}{(4n^2)^2} \sim k^3n^2.$$

Combining the two inequalities, we get an upper bound on k:
$k^3n^2 \lesssim n^4$, or $k \lesssim n^{2/3}$.