Question 1.

- Obtain a planar graph $G'_n$ by removing at most $cr(G_n)$ edges. So $G'_n$ has at least $e(G_n) - cr(G_n)$ edges. Note that $G'_n$ has no triangles, therefore $2e(G'_n) \geq 4f(G'_n)$. By Euler’s formula, we have

$$2 = n - e(G'_n) + f(G'_n) \leq n - \frac{e(G'_n)}{2} \leq n - \frac{e(G_n) - cr(G_n)}{2},$$

or $cr(G_n) \geq e(G_n) - 2n + 4 > 2n$.

- Let $H_p$ be a random subgraph of $G = G_n$ formed by retaining a vertex with probability $p$. An edge $(u, v)$ in $G$ is retained in $H_p$ if both $u$ and $v$ are in $V(H_p)$.

  Since $H_p$ is a subgraph of $G$, it has no triangles, and so by the above we have

$$cr(H_p) > e(H_p) - 2v(H_p).$$

From the linearity of expectation we know

$$\mathbb{E}(cr(H_p)) > \mathbb{E}(e(H_p)) - 2\mathbb{E}(v(H_p)).$$

Note that a crossing survives in $H_p$ with a probability $p^4$. Hence, $\mathbb{E}(cr(H_p)) = p^4 \mathbb{E}(cr(G))$. By definition we have $\mathbb{E}(e(H_p)) = 4np^2$ and $\mathbb{E}(v(H_p)) = pn$. Thus,

$$p^4(e(G)) \geq 4np^2 - 2pn,$$

or

$$cr(G) \geq \frac{4p - 2}{p^3} n.$$

The right hand side is maximized when $p = 3/4$. This gives $cr(G) \geq 2.37n$. 

Question 4.