HW 3

1. The reduction to the case $G_n$ connected with no leaves is straight forward.

Lemma: In such a graph with at most 1 triangle there is a vertex of degree 3.

Pf: As there are no leaves, every edge is involved in 2 faces. As there is at most 1 triangle,

$$2e \geq 3 + 4(f-1) \Rightarrow f \leq \frac{e}{2} + \frac{1}{4} \quad (1)$$

If no vertex had degree 3 or less, then $2e = \sum \deg(v) \geq 4v$. so that $e \geq 2v \quad (2)$. Substituting in Euler's formula gives

$$v - e + f = 2$$
$$v - \frac{e}{2} \geq \frac{3}{2}$$

(2)
$$\frac{e}{2} - \frac{e}{2} = 0 \geq \frac{3}{2}, \text{ contradiction.}$$

So there is a vertex of degree at most 3.

The result will follow from lemma using induction. Let $v \in V(G_n)$ be a vertex of degree at most 3. Then colour $G_n - v$ using induction hypothesis with 4 colours. As $\deg(v) \leq 3$, $v$ may be coloured also.

So $r(G_7) \leq 1$
3. It is possible

4. a) $K_n$ by symmetry.

b) Yes. Let $n$ be such that $c_r(K_n) > |E(K_n)| = \binom{n}{2}$. Such a $n$ will exist as $c_r(K_n) \geq C \cdot \frac{c^2}{n^2} = C \cdot \frac{(\frac{n}{2})^3}{n^2}$ for $C > 0$.

Successively remove edges from $K_n$ in such a way that each removal lowers the crossing number by the smallest number possible. After $(\frac{n}{2})$ removals, we are left with only vertices, so no crossings.

Since $c_r(K_n) > \binom{n}{2}$, it must have been the case that for some graph $G = K_n - E_1, ... E_k$, removing any edge from $G$ lowers the crossing number by at least 2.