HW 3

1. The reduction to the case G_n connected with no leaves is straight forward.

Lemma: In such a graph with at most 1 triangle there is a vertex of degree 3.

Pf: As there are no leaves, every edge is involved in 2 faces. As there is at most 1 triangle,

$$2e \geq 3 + 4(f-1) \Rightarrow f \leq \frac{e}{2} + \frac{1}{4} \qquad (1)$$

If no vertex had degree 3 or less, then $2e = \sum \deg(v_i) \geq 4v$. so that $e \geq 2v$ (2). Substituting in Euler's formula gives

$v - e + f = 2$

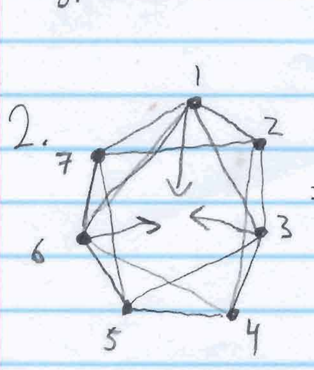
(1) $v - e/2 \geq 3/2$

(2) $e/2 - e/2 = 0 \geq 3/2$, contradiction.

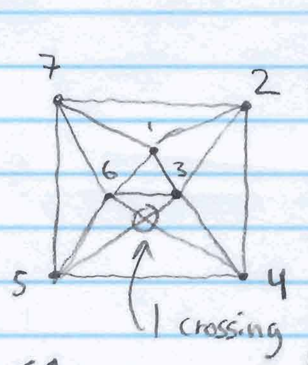
So there is a vertex of degree at most 3.

The result will follow from lemma using induction. Let $v \in V(G_n)$ be a vertex of degree at most 3. Then colour $G_n - v$ using induction hypothesis with 4 colours. As $\deg(v) \leq 3$, v may be coloured also.

2.

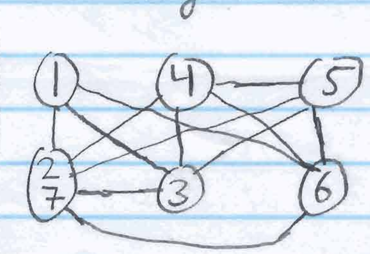


$\Rightarrow$



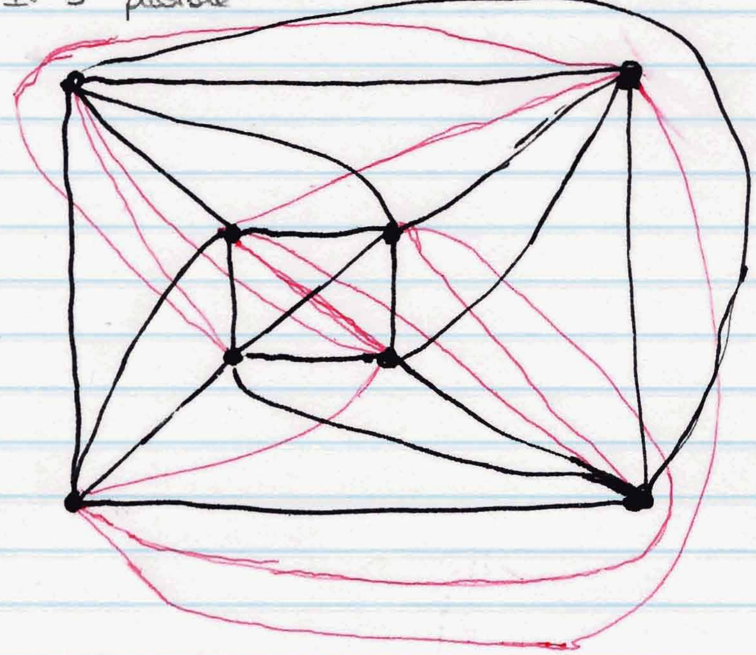
So $Cr(G_7) \leq 1$

Identify 2 and 7 in G_7



Contains $K_{3,3}$ so $Cr(G_7) = 1$

3. It is possible



4. a) K_5 by symmetry.

b) Yes. Let n be such that $Cr(K_n) > |E(K_n)| = \binom{n}{2}$.
Such a n will exist as $cr(K_n) \geq C \cdot \frac{e^3}{n^2} = C \cdot \frac{\binom{n}{2}^3}{n^2}$ for $C > 0$.

- Successively remove edges from K_n in such a way that each removal lowers the crossing number by the smallest number possible. After $\binom{n}{2}$ removals, we are left with only vertices, so no crossings.

Since $Cr(K_n) > \binom{n}{2}$, it must have been the case that for some graph $G := K_n - \{e_1, \dots e_k\}$, removing any edge from G lowers the crossing number by at least 2.