Question 1. Let $G_n$ be a planar graph with at most one triangle. First show that there is a vertex of $G_n$ having degree at most 3:

Suppose not. Then $2e \geq 4n$, where $e$ is the number of edges of $G_n$. Since $G_n$ has only one triangle, we have $2e \geq 3 + 4(f - 1)$, where $f$ is the number of faces (including the infinity face). Now, by Euler’s formula we have

$$2 = n - e + f \leq n - e + \frac{2e + 1}{4} = n - e + \frac{1}{4},$$

and hence $7 + 2e \leq 4n$, contradiction.

Next, we show by induction on $n$ that a planar graph $G_n$ with at most one triangle is 4-colourable:

The base case is trivial. Now, assume that every planar graph with $k$ vertices and at most one triangle is 4-colourable, for some $k$. Suppose we have a planar graph $G_{k+1}$ with $k+1$ vertices and at most one triangle. By the above, there is one vertex $v_0$ having degree at most 3. Remove this vertex (and the edges adjacent to it) from $G_{k+1}$ and obtain a planar graph $G'_k$, which has $k$ vertices and at most one triangle. By induction hypothesis, we can colour $G'_k$ using 4-colours. We can now colour the vertex $v_0$ in $G_{k+1}$ without using the fifth colour because the degree of $v_0$ in $G_{k+1}$ is at most 3. By induction, we are done.

Question 2.

This drawing has one crossing. This graph is non-planar because it contains a minor isomorphic to $K_5$ (by contracting, say, $(2,3)$ and $(3,4)$).
Question 3.

Question 4.
- $K_5$