MATH 309 PRACTICE QUESTIONS.

Note: In the midterm there will be short multiple choice questions and some longer ones like the questions below.

Question 1. Let us suppose that a graph, G_n, has no triangle or C_4. (the smallest cycle has size at least 5)

- What is the maximum number of edges if G_n is planar? (Hint: Use Euler's formula and (double)count the sum of edges in facets, f_i.) Answer:
$$e \leq \frac{5}{3}(n-2)$$
- What is the maximum number of edges if $cr(G_n) = 2$? (Hint: Use the previous answer.) Answer:
$$e \leq \frac{5}{3}(n-2) + 2 = \frac{5n-4}{3}n.$$
- What is the maximum number of edges if $cr(G_n) = n^2$?(Give and approximate answer)

Hint for the last question: Write down the crossing inequality.

Answer:
$$\frac{ce^3}{n^2} \leq cr(G_n) = n^2$$
$$ce^3 \leq n^4$$
$$e \leq c'n^{4/3}.$$

Question 2. Like in the previous question let us suppose that a graph, G_n, has no triangle and C_4. The number of its edges is $10n$. Use the probabilistic method to give a bound on $cr(G_n)$.

Hint: We know from the previous question that if G_m on m vertices has at least $(\frac{5}{3}+\alpha)m$ edges then $cr(G_m) \geq \alpha m$. This holds for any parameter $\alpha > 0$. We choose a probability p such $(\frac{5}{3}+\alpha)pn \sim p^2 10n$. We select the vertices independently at random with probability p. In this random graph the expected crossings number is at least $\alpha pn \leq p^4 cr(G_n)$. The last step is an optimization, find the best value for α.

Answer: After the substitutions from the hint we have $p = \frac{5}{30} + \frac{\alpha}{10}$ and $cr(G_n) \geq \frac{\alpha}{p^3}n$. Solving the optimization problem, finding α which maximizes the right hand side we have $\alpha = \frac{5}{6}$ and then $cr(G_n) \geq \frac{160}{3}n$.

Question 3. Given an arrangement of n lines and n points, L, P, such that most of the lines intersect outside of the convex hull of P. Out of the $\binom{n}{2}$ possible intersections only n line pairs intersect inside the convex hull of P. Give a bound on the number of incidences, $I(P, L)$.

Hint: Draw the geometric graph inside the convex hull as we did in class. On one side you can use the crossing number inequality and for the other side, the "trivial" bound is n instead of $\binom{n}{2.}$

Question 4. A planar graph, G_n, is the edge-disjoint union of two n-vertex trees T_n and P_n.

- Prove that G_n contains a triangle. (Hint: count the number of edges)

  Answer: Both trees contains $n-1$ edges, so their union is a graph with $2n-2$ edges. But as we've seen in class every triangle free planar graph contains at most $2n-4$ edges. So, G_n should contain triangles.
- Prove that the chromatic number of G_n is at most four. (Give a complete proof, don't use the Four Colour Theorem) (Hint: What is the chromatic number of a tree?)

  Answer: Colour the first tree with blue and red and the second by black and white. Now the four colours in G_n are blue-black, blue-white, red-black, and red-white. This is clearly a good colouring of G_n using four colours.

Question 5. A set of curves consists of translates of the log function.

$$C = \{\ln(x-a) + b | a, b \in \mathbb{R}\}.$$

Decide if C is a set of pseudolines. Justify your answer.

Hint: Solve the equation for the intersection point(s)

$$\ln(x-a) + b = \ln(x).$$

Question 6. Do you think it is possible that a planar graph has a triangle but in any planar drawing of it every country has at least four neighbours? Try to draw such a planar graph. Justify your answer.

Answer: Yes, as we've seen in class.