MATH 309 MIDTERM.

1. Write your name and student number on the top of the booklet!

There are four questions to answer.

Question 1. Let us suppose that a planar graph on n vertices has a planar drawing without crossings such that there is a country (facet) with n neighbours (edges).

- Prove that the number of its edges is at most $2n-3$.
- Prove that its chromatic number is at most three.

Solution: We are going to use Euler's formula and (double)count the sum of edges in facets, f_i. So, $2e = \sum_{i=3}^{n} if_i$ and we know that $f_n \geq 1$. It gives the following inequality:

$$2e \geq 3(f-1)+n,$$
$$\frac{2e-n}{3}+1 \geq f.$$

Euler's formula is

$$n-e+f=2$$

and after the substitution

$$e \geq 2n-3.$$

This inequality shows that there is a vertex with degree 3 or less, but that only shows that the chromatic number is at most four. One can prove a better bound by showing that the min. degree is 2, but here we use induction to show that the chromatic number is at most 3.

Base case: The statement is obvious for $n=3$.

Induction: Let us suppose that the statement is true for any graph with less then n vertices. The boundary of the large country is a cycle, C_n. If our graph is just a cycle then it is 3-colourable. If C_n has an edge connecting two vertices then let's separate the graph into two parts along this edge. By the induction hypothesis both parts are 3-colourable and they have only two common vertices, so the whole graph is 3-colourable.

Question 2. Are there non-planar graphs with

- 8 edges? Answer: no, the smallest planar graphs are K_5 with 10 edges and $K_{3,3}$ with 9 edges.

- 11 edges? Yes, add a vertex and an edge to K_5.
- n vertices and $n-1$ edges? Yes, if $n-1 \geq 9$ then draw a $K_{3,3}$ and add the remaining edges arbitrarily.

Justify your answers.

Question 3. Use the probabilistic method to give a good bound on the crossing number of a graph, G_n, which has $10n$ edges.

Answer: For any planar graph, G_m if $e \geq (3+\alpha)m$ then $cr(G_m) \geq \alpha m$. We are going to use this bound in a random subgraph of G_n. We select every vertex with probability p where $(3+\alpha)pn = p^2 10n$. (the expected number of vertices is pn and the expected number of edges is $p^2 e = p^2 10n$.) So, we set $p = \frac{3+\alpha}{10}$. Then for the crossing number we have $p^4 cr(G_n) \geq \alpha pn$.

$$cr(G_n) \geq 10^3 \frac{\alpha}{(3+\alpha)^3} n$$

It was given that the maximum of the function $\frac{x}{(x+3)^3}$ is $\frac{4}{243}$ when $x = \frac{3}{2}$, so we have

$$cr(G_n) \geq \frac{4000}{243} n > 16n.$$

Question 4. A graph on n vertices, G_n, has $n^{4/5}$ edges.

- What is the minimum crossing number of G_n? (Give a lower bound on $cr(G_n)$). Answer: Since $n^{4/5} < n \leq 3n-6$, the graph might be planar with 0 crossing. So the lower bound is 0.
- What is the maximum crossing number? (Give an upper bound on $cr(G_n)$). Answer: The crossing number is obviously smaller than $\binom{e}{2} \sim n^{8/5}/2$, when any pair of edges would intersect. By the crossing inequality for the complete graph on $n^{2/5}$ vertices we have

  $$cr(K_{n^{2/5}}) \geq c \frac{(n^{4/5})^3}{(n^{2/5})^2} = cn^{8/5}$$

  so the sharp upper bound is $cn^{8/5}$.

Note: The hardest question was 1/b. All other problems were repetitions of what we did during the review.