# MATH 309 HW 3. HINTS

There are four questions to answer.

**Question 1.** Let us suppose that a planar graph, $G_n$ has one triangle only and every other cycle is larger. Prove that $\chi(G_n) \leq 4$.

Hint: Use Euler's formula, $v - e + f = 2$. We now that $2e = \sum_{i=3}^{n} i f_i$. So,

$$2e \geq 3 + 4(f - 1) = 4f - 1,$$

which gives the inequality

$$\frac{e}{2} + \frac{1}{4} \geq f.$$

The number of facets is integer, so

$$\frac{e}{2} \geq f.$$

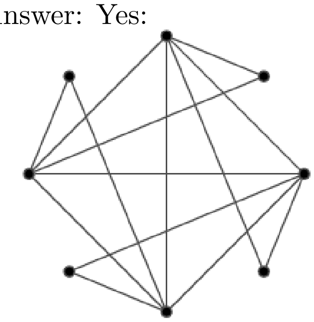
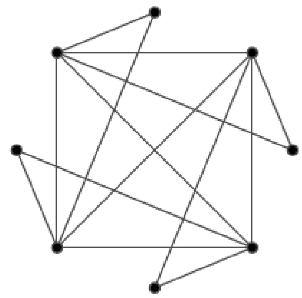
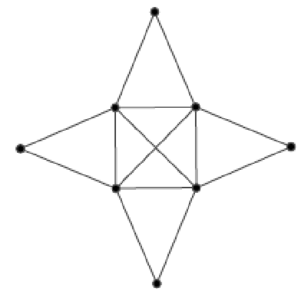
Using Euler's formula one can bound the number of edges which shows that there is always a vertex with degree 3 or less.

**Question 2.** Draw a graph as follows: first draw a regular heptagon (7-gon). The vertices are labelled by numbers $1, 2, \ldots, 7$. The sides of the heptagon give seven edges and now we connect every second vertex with an edge. $(1,3), (3,5), (5,7), (7,2), (2,4), (4,6), (6,1)$. So, this graph has seven vertices and 14 edges. Show that the crossing number of this graph is 1.

Hint: First draw the graph above with one crossing. It shows that the crossing number is 1 or less. To show that the graph is not planar, draw a $K_{3,3}$ inside this graph. (Note that you can glue together two edges to one)

**Question 3.** Can you colour the edges of $K_8$ by blue and red such that both the blue subgraph and the red subgraphs are planar, i.e. both graphs have a way of drawing without crossings?

Answer: Yes:

**Question 4.** In class we noted that every non-planar graph contains an edge so that if we erase this edge then the crossing number of the new graph is smaller.

- Is there a graph such that no matter which edge is deleted the crossing number of the new graph is smaller?
- Is there a graph such that no matter which edge is deleted the crossing number of the new graph reduces by two or more?

Answer: For both questions the answer is yes. For the first one a possible example is $K_5$. Another one is the graph from Question 2.

The second question is much harder. We have to think about the crossing number and the crossing inequality a bit; we are looking for a graph $G_n$ which has an optimal drawing with $cr(G_n)$ crossings where every edge has at least two crossings. Then we know that removing any edge will decrease the crossing number by two, at least. It remains to show that such graph exists. We will prove the existence of such a graph by contradiction. Let us suppose that for every graph and any optimal drawing of it there is an edge, $e$, which has at most one crossing. That would mean that for any graph $G_n$

$$cr(G_n) \leq cr(G_n - e) + 1.$$

It would give the inequality that the crossing number is bounded by the number of edges. (by induction) On the other hand we know from the crossing inequality that e.g.

$$cr(K_n) \geq cn^4 \gg \binom{n}{2},$$

so the previous inequality can't hold for every graph. Done!