A renormalisation group method. IV. Stability analysis

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Abstract

This paper is the fourth in a series devoted to the development of a rigorous renormalisation group method for lattice field theories involving boson fields, fermion fields, or both. The third paper in the series presents a perturbative analysis of a supersymmetric field theory which represents the continuous-time weakly self-avoiding walk on $\mathbb{Z}^d$. We now present an analysis of the relevant interaction functional of the supersymmetric field theory, which permits a nonperturbative analysis to be carried out in the critical dimension $d = 4$. The results in this paper include: proof of stability of the interaction, estimates which enable control of Gaussian expectations involving both boson and fermion fields, estimates which bound the errors in the perturbative analysis, and a crucial contraction estimate to handle irrelevant directions in the flow of the renormalisation group. These results are essential for the analysis of the general renormalisation group step in the fifth paper in the series.

1 Introduction

This paper is the fourth in a series devoted to the development of a rigorous renormalisation group method. The method has been applied to analyse the critical behaviour of the continuous-time weakly self-avoiding walk [4, 5], and the n-component $|\varphi|^4$ spin model [7], in the critical dimension $d = 4$. In both cases, logarithmic corrections to mean-field scaling are established using our method.

In part I [13] of the series, we presented elements of the theory of Gaussian integration and developed norms and norm estimates for performing analysis with Gaussian integrals involving both boson and fermion fields. In part II [14], we defined and analysed a localisation operator whose purpose is to extract relevant and marginal directions in the dynamical system defined by the renormalisation group. In part III [6], we began to apply the formalism of parts I and II to a specific supersymmetric field theory that arises as a representation of the continuous-time weakly self-avoiding walk [5, 12], by studying the flow of coupling constants in a perturbative analysis. We now prove several nonperturbative estimates for the supersymmetric field theory studied in part III. These estimates are essential inputs for our analysis of a general renormalisation group

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step in part V [15], and therefore for the analysis of the critical behaviour of the continuous-time weakly self-avoiding walk in dimension $d = 4$ in [4, 5].

The results in this paper include: proof of stability of the interaction, estimates which enable control of Gaussian expectations involving both boson and fermion fields, estimates which bound the errors in the perturbative analysis of part III and thereby confirm that the perturbative analysis does indeed isolate leading contributions, and a crucial contraction estimate to handle irrelevant directions in the flow of the renormalisation group. All these results are needed in our analysis of a general renormalisation group step in part V. The methods and results developed in this paper are of wider applicability, but for the sake of concreteness, and for the purposes of our specific application in [4, 5], we formulate the discussion in the context of the supersymmetric field theory studied in part III. Supersymmetry is helpful: it ensures that the partition function is equal to 1, so it need never be estimated.

Several mathematically rigorous approaches to renormalisation in statistical mechanics and quantum field theory have been proposed in recent decades, e.g., the books [10, 25, 26]. Characteristic features of the approach we develop are: (i) there is no partition of unity in field space to separate large and small fields, and (ii) fluctuation fields have finite range of dependence. The avoidance of partitions of unity is important for us because it is easier to maintain supersymmetry without them. The use of finite-range fluctuation fields bears some similarity to the wavelet program reviewed in [21], but has better translation invariance properties. An attractive feature of (ii) is that independence of Gaussian fields replaces cluster expansions. The price to be paid for avoiding partitions of unity is that norms must control the size of the basic objects in all of field space, including large fields. The goal of the present paper is to acquire this control.

Our analysis has antecedents in [1, 11, 24], though our setting includes fermions as well as bosons. A systematic development of appropriate norms is given in [13]. Part of the need for these norms is to define complete spaces in order to apply the dynamical system analysis of [8] (discussion of past errors related to completeness can be found in [1]). The norms include a notion that we call "regulators" because they control (regulate) growth when fields are large. These are always a delicate part in this approach and important ideas that guide their choice appear in [19, 20]. For our field theory, the choice of regulators is less delicate because the ϕ^4 term suppresses large fields. Another feature of our analysis is the inclusion of observable fields to permit control of correlation functions; somewhat related ideas were introduced in [18].

The renormalisation group can be defined directly in infinite volume, but until [16] it was not demonstrated that the infinite volume theory defined in this way coincides with the infinite volume defined by taking limits of correlation functions and pressures defined in finite volume. Our analysis also prepares the way for results about this question for the weakly self-avoiding walk.

In the remainder of Section 1, we give the fundamental definitions and provide an informal overview of the results of this paper. The main results are then stated precisely in Section 2. Proofs are given in Sections 3–7. In Appendix A, we prove a lattice Sobolev inequality that lies at the heart of our stability estimates. Finally, Appendix B concerns estimates of a more specialised nature that are required for the analysis of a single renormalisation group step in part V [15]. Our focus throughout the paper is on the case $d = 4$.

1.1 Object of study

We begin with several definitions needed to formulate our results. Many of these definitions are recalled from parts III and I. We begin by introducing the covariance decomposition which provides the basis for a multi-scale analysis. We then introduce the space of boson and fermion fields, and define the interaction functional I. We also recall the definition of the renormalised polynomial V_{pt} from part III, and the definitions of the norms and regulators from part I.

1.1.1 Covariance decomposition

Let $d \geq 4$ and let $\Lambda = \mathbb{Z}^d/L^N\mathbb{Z}$ denote the discrete d-dimensional torus of side L^N, with $L > 1$ fixed (large). We are interested in results which remain useful in the infinite volume limit $N \to \infty$. There are several places in this paper where L must be taken large, depending on unimportant parameters such as the dimension d, or combinatorial constants. We do not comment explicitly on each occasion where L must be taken to be large, but instead *we assume throughout the paper that L is large enough to satisfy each such requirement that is encountered*.

For e in the set $\mathcal{U}$ of $2d$ nearest neighbours of the origin in $\mathbb{Z}^d$, we define the finite difference operator $\nabla^e\phi_x = \phi_{x+e} - \phi_x$, and the Laplacian $\Delta_{\mathbb{Z}^d} = -\frac{1}{2}\sum_{e\in\mathcal{U}} \nabla^{-e}\nabla^e$. Let $C = (-\Delta_\Lambda + m^2)^{-1}$, where $m^2 > 0$ is a positive parameter and Δ_Λ denotes the discrete Laplacian on Λ. We fix N large and m^2 small, and wish to perform an analysis which applies uniformly in N, m^2.

We require decompositions of the covariances $(\Delta_{\mathbb{Z}^d} + m^2)^{-1}$ and $C = (-\Delta_\Lambda + m^2)^{-1}$. For the former, the massless Green function is well-defined for $d > 2$ and we may consider $m^2 \geq 0$, but for the latter we must take $m^2 > 0$. In [6, Section 6.1], there is a detailed discussion of decompositions we use for each of these covariances, based on [3] (see also Section 1.3.1 below). In particular, in [6, Section 6.1] a sequence $(C_j)_{1\leq j<\infty}$ (depending on $m^2 \geq 0$) of positive definite covariances on $\mathbb{Z}^d$ is defined, such that

$$(\Delta_{\mathbb{Z}^d} + m^2)^{-1} = \sum_{j=1}^{\infty} C_j \qquad (m^2 \geq 0). \tag{1.1}$$

The covariances C_j are translation invariant and have the *finite-range* property

$$C_{j;x,y} = 0 \quad \text{if} \quad |x - y| \geq \frac{1}{2}L^j. \tag{1.2}$$

For $j < N$, the covariances C_j can therefore be identified with covariances on Λ, and we use both interpretations. We define

$$w_j = \sum_{i=1}^{j} C_i \qquad (1 \leq j < \infty), \tag{1.3}$$

and note that w_j also obeys (1.2).

There is also a covariance $C_{N,N}$ on Λ such that

$$C = (-\Delta_\Lambda + m^2)^{-1} = \sum_{j=1}^{N-1} C_j + C_{N,N} \qquad (m^2 > 0). \tag{1.4}$$

Thus the finite volume decomposition agrees with the infinite volume decomposition except for the last term in the finite volume decomposition, which is the single term that accounts for the torus.

The expectation $\mathbb{E}_C$ denotes the combined bosonic-fermionic Gaussian integration on $\mathcal{N}$, with covariance C, defined in [13, Section 2.4]. The integral is performed successively, using

$$\mathbb{E}_C = \mathbb{E}_{C_N} \circ \mathbb{E}_{C_{N-1}} \theta \circ \cdots \circ \mathbb{E}_{C_1} \theta, \tag{1.5}$$

where θ defines a type of convolution and is discussed further below.

1.1.2 Fields and field polynomials

We study a field theory which consists of a complex boson field $\phi : \Lambda \to \mathbb{C}$ with its complex conjugate $\bar{\phi}$, a pair of conjugate fermion fields $\psi, \bar{\psi}$, and a *constant* complex observable boson field $\sigma \in \mathbb{C}$ with its complex conjugate $\bar{\sigma}$. The fermion field is given in terms of the 1-forms $d\phi_x$ by $\psi_x = \frac{1}{\sqrt{2\pi i}} d\phi_x$ and $\bar{\psi}_x = \frac{1}{\sqrt{2\pi i}} d\bar{\phi}_x$, where we fix some square root of $2\pi i$. This is the supersymmetric choice discussed in more detail in [13, Sections 2.9–2.10] and used in [6].

Let two points $a, b \in \Lambda$ be fixed. We work with an algebra $\mathcal{N}$ which is defined in terms of a direct sum decomposition

$$\mathcal{N} = \mathcal{N}^{\varnothing} \oplus \mathcal{N}^{a} \oplus \mathcal{N}^{b} \oplus \mathcal{N}^{ab}. \tag{1.6}$$

Elements of $\mathcal{N}^{\varnothing}$ are given by finite linear combinations of products of an even number of fermion fields with coefficients that are functions of the boson fields. This restriction to forms of even degree results in a commutative algebra. Elements of $\mathcal{N}^{a}, \mathcal{N}^{b}, \mathcal{N}^{ab}$ are respectively given by elements of $\mathcal{N}^{\varnothing}$ multiplied by σ, by $\bar{\sigma}$, and by $\sigma\bar{\sigma}$. For example, $\phi_x \bar{\phi}_y \psi_x \bar{\psi}_x \in \mathcal{N}^{\varnothing}$, and $\sigma \bar{\phi}_x \in \mathcal{N}^{a}$. There are canonical projections $\pi_\alpha : \mathcal{N} \to \mathcal{N}^{\alpha}$ for $\alpha \in \{\varnothing, a, b, ab\}$. We use the abbreviation $\pi_* = 1 - \pi_{\varnothing} = \pi_a + \pi_b + \pi_{ab}$. The algebra $\mathcal{N}$ is discussed further around [14, (1.60)] (there $\mathcal{N}$ is written $\mathcal{N}/\mathcal{I}$ but to simplify the notation we write $\mathcal{N}$ here instead). The parameter $p_{\mathcal{N}}$ which appears in its definition is a measure of the smoothness of elements of $\mathcal{N}$ (see [13, Section 2.1]); its precise value is unimportant and can be fixed to be larger than the degree of polynomials encountered in practice in the application of the stability bounds. Constants in estimates may depend on its value, in an unimportant way.

We define the forms

$$\tau_x = \phi_x \bar{\phi}_x + \psi_x \bar{\psi}_x, \qquad \tau_{\nabla\nabla,x} = \frac{1}{2} \sum_{e \in \mathcal{U}} \left((\nabla^e \phi)_x (\nabla^e \bar{\phi})_x + (\nabla^e \psi)_x (\nabla^e \bar{\psi})_x \right), \tag{1.7}$$

$$\tau_{\Delta,x} = \frac{1}{2} \left((-\Delta\phi)_x \bar{\phi}_y + \phi_x (-\Delta\bar{\phi})_y + (-\Delta\psi)_x \bar{\psi}_y + \psi_x (-\Delta\bar{\psi})_y \right). \tag{1.8}$$

Let $\mathcal{Q}$ denote the vector space of polynomials of the form

$$V = V_{\varnothing} + V_a + V_b + V_{ab}, \tag{1.9}$$

where

$$V_{\varnothing} = g\tau^2 + \nu\tau + z\tau_{\Delta} + y\tau_{\nabla\nabla}, \quad V_a = \lambda_a \sigma \bar{\phi}, \quad V_b = \lambda_b \bar{\sigma} \phi, \quad V_{ab} = \lambda_{ab} \bar{\sigma} \sigma, \tag{1.10}$$

$$\lambda_a = -\lambda \mathbb{1}_a, \qquad \lambda_b = -\lambda \mathbb{1}_b, \qquad \lambda_{ab} = -\frac{q}{2}\sigma\bar{\sigma}(\mathbb{1}_a + \mathbb{1}_b), \tag{1.11}$$

$g, \nu, z, \lambda, q \in \mathbb{C}$, and the indicator functions are defined by the Kronecker delta $\mathbb{1}_{a,x} = \delta_{a,x}$. For $X \subset \Lambda$, we write

$$V(X) = \sum_{x \in X} V_x. \tag{1.12}$$

There is an important scale, called the *coalescence scale*, defined by

$$j_{ab} = \big\lfloor \log_L(2|a-b|) \big\rfloor. \tag{1.13}$$

We assume that $\pi_{ab} V = 0$ for $j < j_{ab}$; note that if the coefficient q is initially equal to zero, then under the flow [6, (3.35)] it remains zero below the coalescence scale due to the assumption (1.2).

The goal of our analysis is to understand the Gaussian integral $\mathbb{E}_C e^{-V(\Lambda)}$. Given a positive-definite matrix C whose rows and columns are indexed by Λ, we define the *Laplacian*

$$\mathcal{L}_C = \frac{1}{2}\Delta_C = \sum_{u,v \in \Lambda} C_{u,v} \left(\frac{\partial}{\partial \phi_u} \frac{\partial}{\partial \bar{\phi}_v} + \frac{\partial}{\partial \psi_u} \frac{\partial}{\partial \bar{\psi}_v} \right) \tag{1.14}$$

(see [13, (2.40)]). The Laplacian is intimately related to Gaussian integration. To explain this, suppose we are given an additional boson field $\xi, \bar{\xi}$ and an additional fermion field $\eta, \bar{\eta}$, with $\eta = \frac{1}{\sqrt{2\pi i}} d\xi$, $\bar{\eta} = \frac{1}{\sqrt{2\pi i}} d\bar{\xi}$, and consider the "doubled" algebra $\mathcal{N}(\mathbf{\Lambda} \sqcup \mathbf{\Lambda}')$ containing the original fields and also these additional fields. We define a map $\theta : \mathcal{N}(\mathbf{\Lambda}) \to \mathcal{N}(\mathbf{\Lambda} \sqcup \mathbf{\Lambda}')$ by making the replacement in an element of $\mathcal{N}$ of ϕ by $\phi + \xi$, $\bar{\phi}$ by $\bar{\phi} + \bar{\xi}$, ψ by $\psi + \eta$, and $\bar{\psi}$ by $\bar{\psi} + \bar{\xi}$. According to [13, Proposition 2.6], for a *polynomial* A in the fields, the Gaussian expectation with covariance C can be evaluated using the Laplacian operator via

$$\mathbb{E}_C \theta A = e^{\mathcal{L}_C} A, \tag{1.15}$$

where the fields $\xi, \bar{\xi}, \eta, \bar{\eta}$ are integrated out by $\mathbb{E}_C$, with $\phi, \bar{\phi}, \psi, \bar{\psi}$ kept fixed, and where $e^{\mathcal{L}_C}$ is defined by its power series.

1.1.3 Form of interaction

In [6, Section 2], we discussed reasons to define an interaction

$$I_j(V, \Lambda) = e^{-V(\Lambda)}(1 + W_j(V, \Lambda)), \tag{1.16}$$

where W_j is a certain non-local polynomial in the fields whose definition is recalled below. Our main object of study in this paper is a modified version of I_j which is defined on subsets of Λ.

We recall the relevant definitions from [6]. For polynomials V', V'' in the fields, we define

$$F_C(V', V'') = e^{\mathcal{L}_C}\big(e^{-\mathcal{L}_C} V'\big)\big(e^{-\mathcal{L}_C} V''\big) - V'V'', \tag{1.17}$$

$$F_{\pi,C}(V', V'') = F_C(V', \pi_\varnothing V'') + F_C(\pi_* V', V''). \tag{1.18}$$

By definition, when V' is expanded in $F_C(V', V'')$ as $V' = \pi_\varnothing V' + \pi_* V'$, there are cross-terms $F_C(\pi_\varnothing V', \pi_* V'') + F(\pi_* V', \pi_\varnothing V'')$, and (1.18) is obtained from (1.17) by replacing these cross-terms by $2F_C(\pi_* V', \pi_\varnothing V'')$. This unusual bookkeeping is appropriate (indeed necessary) in the proof of Proposition 4.10.

For nonempty $X \subset \Lambda$, the space $\mathcal{N}(X)$ is defined in [13, (3.38)] as consisting of elements of $\mathcal{N}$ which depend on $\phi_x, \bar{\phi}_x, \psi_x, \bar{\psi}_x$ only with $x \in X$. Recall from [14] that we defined $F \in \mathcal{N}_X$ to mean that there exists a coordinate patch Λ' such that $F \in \mathcal{N}(\Lambda')$ and $X \subset \Lambda'$, and we defined the condition $F \in \mathcal{N}_X$ to guarantee that neither X nor F "wrap around" the torus. The operator $\mathrm{Loc}_X : \mathcal{N}_X \to \mathcal{V}(X)$ is defined in [14, Definition 1.6], and the particular specification we use is that described in [6, Section 3.2]. In particular, the *field dimensions* are $[\phi] = [\bar{\phi}] = [\psi] = [\bar{\psi}] = \frac{d-2}{2}$, and we set $d_+ = d$ on $\mathcal{N}^{\varnothing}$, with a subtler choice of d_+ on each of $\mathcal{N}^a, \mathcal{N}^b, \mathcal{N}^{ab}$.

For $x \in \Lambda$, with w_j given by (1.3) we define

$$W_j(V, x) = \frac{1}{2}(1 - \mathrm{Loc}_x) F_{\pi, w_j}(V_x, V(\Lambda)) \qquad (j < N). \tag{1.19}$$

For $j < N$, the above application of Loc_x is well-defined since $F_{\pi,w_j}(V_x, V(\Lambda)) \in \mathcal{N}_x$ due to the finite-range property of w_j. For $X \subset \Lambda$, we then define

$$W_j(V, X) = \sum_{x \in X} W_j(V, x). \tag{1.20}$$

By definition, $w_0 = 0$ and $W_0 = 0$.

We consider the natural paving of Λ by disjoint blocks of side length L^j, for $j = 0, \dots, N$. The set of all scale-j blocks is denoted $\mathcal{B}_j$, and $\mathcal{P}_j$ denotes the set whose elements are finite unions of blocks in $\mathcal{B}_j$. We refer to elements of $\mathcal{P}_j$ as scale-j *polymers*. Given a polynomial $V \in \mathcal{V}$, and $X \subset \Lambda$, let

$$\mathcal{I}(V, X) = e^{-V(X)}. \tag{1.21}$$

The interaction is defined, for $B \in \mathcal{B}_j$ and $X \in \mathcal{P}_j$, by

$$I_j(V, B) = \mathcal{I}(V, B)\,(1 + W_j(V, B))\,, \qquad I_j(V, X) = \prod_{B \in \mathcal{B}_j(X)} I_j(V, B). \tag{1.22}$$

Due to the finite-range property (1.2), $I_j(V, B) \in \mathcal{N}(B^+)$, where B^+ denotes the union of B with every block B' such that $B \cup B'$ is connected. We often write $I_j(V, X) = I_j^X(V)$. We also consider the interaction defined, for $b \in \mathcal{B}_{j-1}$ and $X \in \mathcal{P}_{j-1}$, by

$$\tilde{I}_j(V, b) = \mathcal{I}(V, b)(1 + W_j(V, b)), \qquad \tilde{I}_j(V, X) = \prod_{b \in \mathcal{B}_{j-1}(X)} \tilde{I}_j(V, b). \tag{1.23}$$

Thus $\tilde{I}_j$ is defined on blocks and polymers of scale $j - 1$, whereas I_j is defined on blocks and polymers of scale j.

An element $F \in \mathcal{N}$ is said to be *gauge invariant* if it is invariant under the gauge flow $q \mapsto e^{-2\pi i t} q$, $\bar{q} \mapsto e^{+2\pi i t} \bar{q}$; for all $q = \phi_x, \psi_x, \sigma$; $\bar{q} = \bar{\phi}_x, \bar{\psi}_x, \bar{\sigma}$; and $x \in \Lambda$. The basic objects we study, including $V, F, W, I, \tilde{I}$, are all gauge invariant. Also, since we assume $V_{ab} = 0$ for $j < j_{ab}$, it follows that none of these basic objects has a nonzero component in $\mathcal{N}_{ab}$ unless $j \geq j_{ab}$.

1.1.4 The renormalised field polynomial

To simplify the notation, we write $\mathcal{L}_j = \mathcal{L}_{C_j}$. Given $V \in \mathcal{Q}$, as in [6, (3.22)] we define

$$P_j(V, x) = \mathrm{Loc}_x \left(e^{\mathcal{L}_{j+1}} W_j(V, x) + \frac{1}{2} F_{\pi, C_{j+1}}(e^{\mathcal{L}_{j+1}} V_x, e^{\mathcal{L}_{j+1}} V(\Lambda)) \right) \quad (j + 1 < N), \tag{1.24}$$

and we write $P_j(V,X) = \sum_{x\in X} P_j(V,x)$ for $X \subset \Lambda$. The local polynomial $V_{\rm pt}$ is defined, as in [6, (3.23)], by

$$V_{{\rm pt},j+1}(V,x) = e^{\mathcal{L}_{j+1}} V_x - P_j(V,x) \quad (j+1<N). \tag{1.25}$$

By the finite-range property (1.2), if $B \in \mathcal{B}_{j+1}$ then $V_{{\rm pt},j+1}(V,B) \in \mathcal{N}(B^+)$, with B^+ the enlargement of B defined below (1.22). By [6, Lemma 5.2] we have $e^{\mathcal{L}_{j+1}} V = V + 2gC_{j+1;0,0}\tau$, so

$$V_{{\rm pt},j+1} = V + 2gC_{j+1;0,0}\tau - P_j(V) \quad (j+1<N). \tag{1.26}$$

For $j<N$, an explicit formula $V_{{\rm pt},j} = \varphi_{{\rm pt},j-1}$ is given in [6, Proposition 4.1]. In particular, $P \in \mathcal{Q}$. The definition of $V_{\rm pt}$ is motivated by the fact (shown in [6, Section 2]) that the definitions of W and $V_{\rm pt}$ cooperate to arrange that, as formal power series,

$$\mathbb{E}\theta I_j(V,\Lambda) \approx I_{j+1}(V_{\rm pt},\Lambda) + O(V^3). \tag{1.27}$$

For $B \in \mathcal{B}_j$, we make the abbreviation

$$\tilde{I}_{\rm pt}(B) = \tilde{I}_{j+1}(V_{\rm pt}, B), \tag{1.28}$$

1.1.5 The final scale

The above definitions have been given for scales below but not including the final scale N. At scale N, the torus consists of a single block $\Lambda \in \mathcal{B}_N$, the periodicity of the torus becomes preponderant, the definition of ${\rm Loc}_x F_{\pi,w_{N,N}}(V_x, V(\Lambda))$ breaks down due to lack of a coordinate patch, and the definitions of W and P in (1.19) and (1.24) can no longer be used. Initially this may appear problematic, since we are ultimately interested in performing the last expectation and computing I_N. However, any apparent difficulty is only superficial. There is only one problematic scale out of an unbounded number of scales. Moveover, the covariance $C_{N,N}$ is extremely small for large m^2L^{2N} (see (1.70) below), and we do take the limit $N\to\infty$ before $m^2 \downarrow 0$, so the last expectation is insignificant. Nevertheless it is necessary to make appropriate definitions of $V_{\rm pt}$ and W at scale N. We do this in such a way that the analysis at scale N differs minimally from that at previous scales.

For $V_{\rm pt}$, the natural choice $V_{{\rm pt},N} = \varphi_{{\rm pt},N-1}$ is made in [6, Definition 4.2]; this choice defines $V_{{\rm pt},N}$ to be equal to what it would be if the torus side length were at a higher scale than scale N. In terms of this choice, we define P_{N-1} so that (1.25) remains valid for scale N, namely

$$P_{N-1}(V) = -V_{{\rm pt},N}(V) + \mathbb{E}_{C_N}\theta V. \tag{1.29}$$

There is no P_N, the last P_j is P_{N-1} since the last $V_{\rm pt}$ is $V_{{\rm pt},N}$. Thus we have arranged the definitions at the last scale in such a way that $V_{{\rm pt},N}$ agrees with what it would be on a torus of scale greater than N (the use of $\mathbb{E}_{C_N}$ rather than $\mathbb{E}_{C_{N,N}}$ is intentional and for this reason).

For W_N, our choice is inspired by a key identity obeyed by W_j for $j<N$, proved in Lemma 4.5. The identity implies in particular that

$$W_j(V,x) = e^{\mathcal{L}_j} W_{j-1}(e^{-\mathcal{L}_j}V, x) - P_{j-1}(e^{-\mathcal{L}_j}V, x) + \frac{1}{2}F_{\pi,C_j}(V_x, V(\Lambda)) \quad (j<N). \tag{1.30}$$

The above identity is instrumental in the proof that the perturbative analysis of [6] is accurate beyond formal power series, and thus plays a fundamental role. We define W_N to maintain this identity. Thus, with P_{N-1} given by (1.29), we define

$$W_N(V,x) = e^{\mathcal{L}_{N,N}} W_{N-1}(e^{-\mathcal{L}_{N,N}}V, x) - P_{N-1}(e^{-\mathcal{L}_{N,N}}V, x) + \frac{1}{2}F_{\pi,C_{N,N}}(V_x, V(\Lambda)). \tag{1.31}$$

1.1.6 Norms and field regulators

Our estimates are typically expressed in terms of the T_ϕ semi-norm and two important functions of ϕ that we refer to as *field regulators*. We now recall the relevant definitions.

The T_ϕ semi-norm

We make heavy use of the $\Phi_j(\mathfrak{h}_j)$ norm on test functions and the $T_{\phi,j}(\mathfrak{h}_j)$ semi-norm on $\mathcal{N}$. The definition of the $\Phi_j(\mathfrak{h}_j)$ norm on test functions is given in [13, Example 3.2] in terms of a parameter $p_\Phi \geq d+1-\frac{d-2}{2} = \frac{d+4}{2}$ (consistent with the requirement above the statement of [14, Proposition 1.12]), and here we take $R = L^j$ in [13, Example 3.2] where j is the scale. The value of p_Φ is fixed but unimportant, and constants in estimates may depend on it. The space $\Phi(\mathfrak{h})$ consists of test functions $g : \vec{\mathbf{\Lambda}}^* \to \mathbb{C}$. The definition of the norm requires the specification of its "sheets" and the values of the components of $\mathfrak{h}_j$ for each sheet (particular choices are made in Section 1.3.2 below). We assume that in the definition of the norm there are sheets for each of the fields $\phi, \bar\phi, \psi, \bar\psi, \sigma, \bar\sigma$. The boson and fermion fields have a common component of $\mathfrak{h}_j$, and we sometimes abuse notation by writing $\mathfrak{h}_j$ for this particular component value. Also, the fields $\sigma, \bar\sigma$ have a common value $\mathfrak{h}_{\sigma,j}$.

The $T_\phi(\mathfrak{h})$ semi-norm is defined in [13, Definition 3.3], and provides a family of semi-norms indexed by the vector $\mathfrak{h}$. We often keep $\mathfrak{h}$ as a parameter in our results, as our applications ultimately use more than one choice. Properties of the T_ϕ semi-norm are derived in [13]; prominent among them is the product property of [13, Proposition 3.7] which asserts that $\|FG\|_{T_\phi} \leq \|F\|_{T_\phi}\|G\|_{T_\phi}$ for all $F, G \in \mathcal{N}$.

Fluctuation-field regulator

A special case of the $\Phi(\mathfrak{h})$ norm is obtained by regarding the boson field as a test function: given $\mathfrak{h}_j > 0$ its $\Phi_j = \Phi_j(\mathfrak{h}_j)$ norm is

$$\|\phi\|_{\Phi_j(\mathfrak{h}_j)} = \mathfrak{h}_j^{-1} \sup_{x\in\Lambda} \sup_{|\alpha|_1 \leq p_\Phi} L^{j|\alpha|_1} |\nabla^\alpha \phi_x|. \tag{1.32}$$

The estimates given in [6, Proposition 6.1] (see [6, (6.102)]) for the covariance decomposition show, in particular, that

$$|\nabla_x^\alpha \nabla_y^\beta C_{j;x,y}| \leq c L^{-(j-1)(2[\phi]+(|\alpha|_1+|\beta|_1))}. \tag{1.33}$$

with $[\phi]$ the *field dimension*

$$[\phi] = \frac{d-2}{2} \tag{1.34}$$

and where c is independent of j, L and $m^2 \in [0,\delta]$ for $j < N$, while in the special case $C_j = C_{N,N}$, c is independent of N, L, m^2 as long as $m^2 \in [\varepsilon L^{-2(N-1)}, \delta]$ with the constant c now depending on $\varepsilon > 0$. This suggests that under the expectation $\mathbb{E}_{C_j}$, $|\nabla^\alpha \phi_x|$ is typically $O(L^{-(j-1)([\phi]+(|\alpha|_1))})$. We choose a value ℓ_j for $\mathfrak{h}_j$ which makes the norm $\|\phi\|_{\Phi_j(\ell_j)}$ be small for typical ϕ, i.e., we choose for $\mathfrak{h}_j$ the value

$$\ell_j = \ell_0 L^{-j[\phi]}, \tag{1.35}$$

with an L-dependent (large) constant ℓ_0 whose value gets fixed at (1.72) below.

As in [13, (3.37)], for $X \subset \Lambda$ we define a local norm of the boson field ϕ by

$$\|\phi\|_{\Phi_j(X)} = \inf\{\|\phi - f\|_{\Phi_j} : f \in \mathbb{C}^\Lambda \text{ such that } f_x = 0 \ \forall x \in X\}. \tag{1.36}$$

This definition localises the norm to X by minimising over all extensions to the complement of X. A *small set* is defined to be a connected polymer $X \in \mathcal{P}_j$ consisting of at most 2^d blocks (the specific number 2^d plays a role only in the combinatorial geometry of [15, Section C] and it is only important in this paper that it be a finite constant independent of L). The set of small sets is denoted $\mathcal{S}_j \subset \mathcal{P}_j$. The *small set neighbourhood* of $X \subset \Lambda$ is the enlargement of X defined by

$$X^\square = \bigcup_{Y \in \mathcal{S}_j : X \cap Y \neq \varnothing} Y. \tag{1.37}$$

Given $X \subset \Lambda$ and $\phi \in \mathbb{C}^\Lambda$, we recall from [13, Definition 3.14] that the *fluctuation-field regulator* G_j is defined by

$$G_j(X, \phi) = \prod_{x \in X} \exp\left(|B_x|^{-1} \|\phi\|^2_{\Phi_j(B_x^\square, \ell_j)} \right), \tag{1.38}$$

where $B_x \in \mathcal{B}_j$ is the unique block that contains x, and hence $|B_x| = L^{dj}$.

Large-field regulator

For $j < N$ (and L large), and for $B \in \mathcal{B}_j$, the diameter of $B^\square$ is less than the period of the torus. We can therefore identify $B^\square$ with a subset of $\mathbb{Z}^d$ and use this identification to define polynomial functions from $B^\square$ to $\mathbb{C}$. We define

$$\widetilde{\Pi}(B^\square) = \left\{ f \in \mathbb{C}^\Lambda \mid f \text{ restricted to } B^\square \text{ is a linear polynomial } \right\}. \tag{1.39}$$

Then, for $\phi \in \mathbb{C}^\Lambda$, we define the semi-norm

$$\|\phi\|_{\tilde{\Phi}(B^\square)} = \inf\{\|\phi - f\|_\Phi : f \in \widetilde{\Pi}(B^\square)\}. \tag{1.40}$$

We recall from [13, Definition 3.15] that the *large-field regulator* $\tilde{G}_j$ is defined by

$$\tilde{G}_j(X, \phi) = \prod_{x \in X} \exp\left(\frac{1}{2} |B_x|^{-1} \|\phi\|^2_{\tilde{\Phi}_j(B_x^\square, \ell_j)} \right). \tag{1.41}$$

The definition (1.41) is only used for $j < N$, since the norm on its right-hand side is not defined at the final scale $j = N$. Since $\|\phi\|_{\tilde{\Phi}(B^\square)} \leq \|\phi\|_{\Phi(B^\square)}$ by definition, $\tilde{G}_j(X, \phi) \leq G_j(X, \phi)^{1/2}$. The $\frac{1}{2}$ in the exponent of (1.41) is a convenience that was used in [13, Proposition 3.17]. The role of $\tilde{G}_j$ is discussed in Section 1.2.1 below.

Regulator norms

The two regulators lead us to the following definition.

Definition 1.1. Norms on $\mathcal{N}(X^\square)$ are defined, for $F \in \mathcal{N}(X^\square)$ and $\gamma \in (0,1]$, by

$$\|F\|_{G_j} = \sup_{\phi \in \mathbb{C}^\Lambda} \frac{\|F\|_{T_{\phi,j}}}{G_j(X,\phi)} \quad j \le N, \tag{1.42}$$

$$\|F\|_{\tilde{G}_j^\gamma} = \sup_{\phi \in \mathbb{C}^\Lambda} \frac{\|F\|_{T_{\phi,j}}}{\tilde{G}_j^\gamma(X,\phi)} \quad j < N. \tag{1.43}$$

The norms depend on the choice of $\mathfrak{h}_j$ used in the $T_{\phi,j}(\mathfrak{h}_j)$ semi-norm on the right-hand sides. We write $\|F\|_j$ for the left-hand sides of (1.42)–(1.43) in statements that apply to both the G and $\tilde{G}$ norms. Note that the norm $\|F\|_{G_j}$ is defined for all scales $j \le N$ whereas we $\|F\|_{\tilde{G}_j}$ is undefined at the last scale. At scale N, statements about the norm $\|F\|_j$ are to be understood as applying *only* to the G norm.

A fundamental property of the norms (1.42)–(1.43) is that each obeys the *product property*

$$\|FG\|_j \le \|F\|_j \|G\|_j \qquad \text{when } F \in \mathcal{N}(X), G \in \mathcal{N}(Y) \text{ for } \textit{disjoint } X, Y \in \mathcal{P}_j. \tag{1.44}$$

This is an immediate consequence of the above mentioned product property which states that $\|FG\|_{T_\phi} \le \|F\|_{T_\phi} \|G\|_{T_\phi}$ for *any* $F, G \in \mathcal{N}$, together with the fact that by definition $G_j(X \cup Y, \phi) = G_j(X,\phi) G_j(Y,\phi)$ for disjoint X, Y, and similarly for $\tilde{G}_j$.

1.2 Overview of results

Our goal in this paper is to obtain a thorough understanding of the interaction functional $I = I_j$. The main results are stated in Section 2, with proofs deferred to Sections 3–7. The results include proof of stability bounds for I, estimates on Gaussian expectations involving both boson and fermion fields, estimates verifying the accuracy of the perturbative calculations in [6], and proof of the crucial contraction property needed to control irrelevant directions in the flow of the renormalisation group. These all play a role in the analysis of a single renormalisation group step in [15]. Before making precise statements in Section 2, in this section we provide an informal overview of and motivation for the results.

1.2.1 Stability, expectation and the large-field problem

In Section 2.1, we state a series of *stability estimates*. In particular, Proposition 2.1 provides the bound

$$\|I_j(V,B)F(B)\|_{T_\phi(\ell_j)} \le 2\|F(B)\|_{T_0(\ell_j)} G_j(B,\phi) \tag{1.45}$$

for $B \in \mathcal{B}_j$, and for a polynomial $F(B)$ in the fields in B of degree at most the parameter $p_{\mathcal{N}}$ in the definition of the space $\mathcal{N}$, under suitable hypotheses expressing a smallness condition on the coupling constants in V. Since $G_j(B,\phi) = \exp[\|\phi\|^2_{\Phi(B^\square,\ell_j)}]$, (1.45) provides information on the growth of the left-hand side for large fields ϕ. This estimate does not take advantage of the quartic decay provided by $e^{-g\tau^2}$ to compensate for the quadratic part $e^{-\nu\tau}$ in e^{-V} (with ν possibly negative). This is reflected by the quadratic growth in the exponent on the right-hand side of (1.45).

The renormalisation group method is based on iterated expectation to progressively take into account fluctuations on increasingly larger scales. One difficulty with (1.45) is that it degenerates

under expectation and change of scale, as we discuss next. These ideas play a role in the proof of Proposition 2.7, which is our main estimate on Gaussian expectation. We make the abbreviation $\mathbb{E}_j = \mathbb{E}_{C_j}$ for the Gaussian expectation with covariance C_j. Since the expectation involves both boson and fermion fields (see [12, 13]), it would more accurately be termed "super-expectation" but we use the term "expectation" for brevity. It is shown in [13, Proposition 3.19], that for any $K \in \mathcal{N}$,

$$\|\mathbb{E}_{j+1}\theta K\|_{T_\phi(\mathfrak{h}_j)} \le \mathbb{E}_{j+1}\|K\|_{T_{\phi\sqcup\xi}(\mathfrak{h}_j\sqcup\ell_j)}. \tag{1.46}$$

In more detail, in [13, Proposition 3.19] we choose $w = \mathfrak{h}_j$ and $w' = \ell_j$, and the hypothesis $\|C_{j+1}\|_{\Phi_{j+1}(\ell_{j+1})} \le 1$ is verified at (1.72) below. The integrand on the right-hand side of (1.46) is a function only of the boson field, so the super-expectation reduces to a standard Gaussian expectation with covariance C_{j+1} (see [13, Section 2.8]). The fermion field ceases to play a significant role in the analysis once this inequality has been applied, and this is a beneficial aspect of our method.

By (1.45)–(1.46) and (1.38), and by the inequality $\|\phi+\xi\|^2 \le 2(\|\phi\|^2 + \|\xi\|^2)$,

$$\|\mathbb{E}_{j+1}\theta I_j(V,B)\|_{T_\phi(\ell_j)} \le \mathbb{E}_{j+1}\|I_j(V,B)\|_{T_{\phi\sqcup\xi}(\ell_j\sqcup\ell_j)} \le 2G_j(B,\phi)^2\mathbb{E}_{j+1}G_j(B,\xi)^2. \tag{1.47}$$

According to [13, Proposition 3.20], $\mathbb{E}_{j+1}G_j(B,\xi)^2 \le 2$. Therefore,

$$\|\mathbb{E}_{j+1}\theta I_j(V,B)\|_{T_\phi(\ell_j)} \le 4G_j(B,\phi)^2. \tag{1.48}$$

The left-hand side can only become smaller when the semi-norm is changed from scale j to scale $j+1$ (this useful monotonicity property is proved in Lemma 3.2 below). To see the effect of a change of scale on the right-hand side, consider the particular case $\phi_x = a$ for all x, where a is a constant. In this case, by definition,

$$\begin{aligned} L^{-dj}\|\phi\|^2_{\Phi_j(B^\square_{x,j},\ell_j)} &= L^{-dj}\ell_j^{-2}a^2 = L^2L^{-d(j+1)}\ell_{j+1}^{-2}\,a^2 = L^2L^{-d(j+1)}\|\phi\|^2_{\Phi_{j+1}(B^\square_{x,j},\ell_{j+1})} \\ &= L^2L^{-d(j+1)}\|\phi\|^2_{\Phi_{j+1}(B^\square_{x,j+1},\ell_{j+1})}, \end{aligned} \tag{1.49}$$

so for this case $G_j(B,a) = G_{j+1}^{L^2}(B,a)$. Thus the estimate after expectation and change of scale is substantially worse than (1.45) (it is the growth in ϕ that is problematic, the constant 4 in (1.48) is not). It is in this way that the so-called *large-field problem* enters our analysis. We postpone the problem by setting $\phi = 0$, so that the regulator plays no role in (1.48). With $\phi = 0$, (1.47) becomes

$$\|\mathbb{E}_{j+1}\theta I_j(V,B)\|_{T_0(\ell_j)} \le \mathbb{E}_{j+1}\|I_j(V,B)\|_{T_{0\sqcup\xi}(\ell_j\sqcup\ell_j)} \le 4. \tag{1.50}$$

From this, we see that control of I_j is needed for *all* field values in order to estimate the expectation of the fluctuation field ξ, even when $\phi = 0$. Thus we are able to obtain a useful estimate in the T_0 semi-norm at scale $j+1$, but this is not sufficient to be able to iterate these estimates as the scale advances.

To deal with the large-field problem, we do not perform a separate analysis on regions of space where the field is large and where it is small, as has been done in other renormalisation group methods, e.g., [2, 17, 22, 23]. Instead, we take advantage of the factor $e^{-g\sum_{x\in B}|\phi_x|^4}$ in $I(B)$ and exploit it to capture the notion that a typical field should roughly have size $g^{-1/4}L^{-jd/4}$. For this, we need information about the size of g.

Our ansatz is that at scale j, g is close in size to $\bar{g}_j$, which is defined by the recursion

$$\bar{g}_{j+1} = \bar{g}_j - \beta_j\bar{g}_j^2 \tag{1.51}$$

of [6, (4.13)], with a fixed initial condition $\bar{g}_0$, and with β_j given in terms of the covariance w_j of (1.3) by

$$\beta_j = 8 \sum_{x \in \Lambda} (w_{j+1;0,x}^2 - w_{j;0,x}^2). \tag{1.52}$$

The sequence β_j is closely related to the *bubble diagram* $\sum_{x \in \mathbb{Z}^d} [(-\Delta_{\mathbb{Z}^d}^{-1})_{0x}]^2$, which diverges for $d = 4$ but converges for $d > 4$ since the inverse Laplacian is asymptotically a multiple of $|x|^{-(d-2)}$. By [6, Lemma 6.3], $\beta_j \to 0$ for $d > 4$ whereas $\beta_j \to \pi^{-2} \log L$ for $d = 4$. Also, by choosing $\bar{g}_0$ to be sufficiently small, it follows that $\bar{g}_j$ is uniformly small.

In the present paper, our focus is on the advancement of one scale to the next, rather than on all scales simultaneously. Because of this, and to provide flexibility, rather than using $\bar{g}_j$, we introduce a small positive $\tilde{g}_j$ and consider g at scale j to be close to $\tilde{g}_j$. We do not assume that $\tilde{g}_j$ is given by (1.51) (a different but closely related choice of $\tilde{g}$ is used in [5, (6.15)]), but we do assume that $\tilde{g}_j$ is uniformly small for all j, and that we are free to choose how small it is depending on L. Thus we introduce $h_j \propto \tilde{g}_j^{-1/4} L^{-jd/4}$ and seek estimates in terms of the $T_\phi(h_j)$ semi-norm. Note that for $d = 4$, h_j is larger than ℓ_j by a factor $\tilde{g}_j^{-1/4}$.

We employ the $T_\phi(h_j)$ semi-norm in conjunction with the large-field regulator $\tilde{G}_j$. An essential property of $\tilde{G}_j$ (used in the proofs of Propositions 2.2–2.3 and 2.7–2.8 below) is given in the following lemma. We apply Lemma 1.2 with specific choices of p, and do not thereby lose control of the size of L.

Lemma 1.2. *Let $X \subset \Lambda$. For any fixed $p > 0$ (no matter how large), if L is large enough depending on p, then for all $j + 1 < N$,*

$$\tilde{G}_j(X, \phi)^p \le \tilde{G}_{j+1}(X, \phi). \tag{1.53}$$

Proof. Let $d_+ = [\phi] + 1 = \frac{d-2}{2} + 1 = \frac{d}{2}$. According to the definition of dimension of a polynomial given in [14, Section 1.3], a linear polynomial has dimension $[\phi] + 1 = d_+$. It is a consequence of [14, Lemma 3.6], with $d'_+ = d_+ + 1 = \frac{d}{2} + 1$, that

$$\|\phi\|_{\tilde{\Phi}_j(B_{j,x}^\square, \ell_j)} \le c L^{-d/2-1} \|\phi\|_{\tilde{\Phi}_{j+1}(B_{j,x}^\square, \ell_{j+1})}. \tag{1.54}$$

Therefore, since the semi-norm (1.40) is non-decreasing in X by definition,

$$L^{-jd} \sum_{x \in X} \|\phi\|_{\tilde{\Phi}_j(B_{j,x}^\square, \ell_j)}^2 \le c L^{-2} L^{-(j+1)d} \sum_{x \in X} \|\phi\|_{\tilde{\Phi}_{j+1}(B_{j+1,x}^\square, \ell_{j+1})}^2, \tag{1.55}$$

from which (1.53) follows when L is large enough that $pcL^{-2} \le 1$. ∎

The inequality (1.53) does not hold for the regulator G: we have concluded from (1.49) that for a constant field we have $G_j = G_{j+1}^{L^2}$. In contrast, the norm in $\tilde{G}$ scales down, because it does not examine the constant and linear parts of ϕ. By the use of a lattice Sobolev inequality (proved in Appendix A), we take advantage of the decay in $e^{-g\tau^2}$ to cancel the exponential quadratic $\|\phi\|_\Phi^2$ at the cost of an exponential of $\|\phi\|_{\tilde{\Phi}}^2$. By pursuing this strategy, we prove in Proposition 2.1 below that for $F(B)$ as in (1.45),

$$\|I_j(V, B) F(B)\|_{T_\phi(h_j)} \le 2 \|F(B)\|_{T_0(h_j)} \tilde{G}_j(B, \phi), \tag{1.56}$$

and now with (1.53) this leads as above to

$$\|\mathbb{E}_{j+1}\theta I_j(V,B)F(B)\|_{T_\phi(h_j)} \le 4\|F(B)\|_{T_0(h_j)}\tilde{G}_j(B,\phi)^2 \le 4\|F(B)\|_{T_0(h_j)}\tilde{G}^\gamma_{j+1}(B,\phi), \tag{1.57}$$

for any fixed choice of $\gamma \in (0,1]$, e.g., $\gamma = 1/2$, with L large depending on γ. Thus the h bound reproduces itself after expectation and change of scale. In fact, our ability to choose $\gamma < 1$ shows that the h bound *improves*.

On the other hand, the ℓ bound degrades after expectation and change of scale. However, together the scale-$(j+1)$ ℓ and h bounds can be combined using [13, Proposition 3.17] to infer a G_{j+1} bound for all ϕ from the $T_0(\ell_{j+1})$ and $\tilde{G}_{j+1}$ bounds. In this way it is possible to obtain bounds at scale $j+1$ of the same form as the bounds at scale j. We postpone the application of [13, Proposition 3.17] to the proof of [15, Theorem 1.11]. With this motivation, throughout this paper we prove estimates in terms of the two norm pairs

$$\|F\|_j = \|F\|_{G_j(\ell_j)} \quad \text{and} \quad \|F\|_{j+1} = \|F\|_{T_{0,j+1}(\ell_{j+1})}, \tag{1.58}$$

and

$$\|F\|_j = \|F\|_{\tilde{G}_j(h_j)} \quad \text{and} \quad \|F\|_{j+1} = \|F\|_{\tilde{G}^\gamma_{j+1}(h_{j+1})}, \tag{1.59}$$

i.e., estimates on $\|F\|_{j+1}$ are expressed in terms of $\|F\|_j$ for each of the pairs (1.58) and (1.59). We distinguish the cases (1.58) and (1.59) by writing $\mathfrak{h}_j = \ell_j$ to indicate (1.58), and $\mathfrak{h}_j = h_j$ to indicate (1.59).

Iteration of estimates using (1.59) is possible without the accompaniment of (1.58). However, estimates in terms of the $\tilde{G}(h)$ norm are insufficient on their own to make estimates on remainder terms in the flow of coupling constants, and without such estimates we are unable to study critical behaviour. In the flow of coupling constants determined in [15], the interaction polynomial V_{j+1} at scale $j+1$ is expressed in terms of $V_{\mathrm{pt},j+1}(V_j)$ plus a non-perturbative remainder $\rho_{j+1} \in \mathcal{Q}$ whose coupling constants must be shown to be third order in $\tilde{g}_j$. Our control over these coupling constants is obtained via the T_0 semi-norm. To illustrate this, consider the case of $d = 4$, and suppose that the τ^2 term in ρ_{j+1} were simply $\tilde{g}_j^3\tau^2$. The calculation of the T_ϕ semi-norm of τ^2 is straightforward, and a small extension of [13, Proposition 3.9] gives $\|\tilde{g}_j^3\tau_x^2\|_{T_0(\mathfrak{h}_j)} \asymp \tilde{g}_j^3\mathfrak{h}_j^4$. Focussing only on the power of $\tilde{g}_j$, the choice $\mathfrak{h}_j = h_j$ gives an overall power $\tilde{g}_j^3(\tilde{g}_j^{-1/4})^4 = \tilde{g}_j^2$, which is second order rather than the desired third order. For this reason, estimates in terms of norms with $\mathfrak{h} = h$ are insufficient. On the other hand, with the $T_0(\ell)$ semi-norm there is no loss of powers of $\tilde{g}_j$ arising from $\|\tau_x^2\|_{T_0(\ell_j)} \asymp \ell_j^4$, and the $T_0(\ell)$ semi-norm indeed identifies $\tilde{g}_j^3\tau^2$ as a third-order term.

Remark 1.3. The advancement of estimates to the final scale N is special, since the $\tilde{G}$ norm is undefined at that scale. However, the work of the $\tilde{G}$ norm is complete by scale N, as there is no further difficulty concerning degradation of estimates since the scale no longer advances. Thus, at scale N, we can consider the norm to be the G norm with regulator G replaced by a suitable large power of G_{N-1}, such as G_{N-1}^{10} (using G_{N-1}^2 would be sufficient for (1.48) but higher powers are required later). Then a scale-N estimate $\|F\|_N \le C$ is interpreted as stating that $\|F\|_{T_\phi,N} \le CG_{N-1}(\Lambda,\phi)^{10}$. In some applications, the T_0 estimate obtained by setting $\phi = 0$ is sufficient. More generally, the estimate states that $\|F\|_{T_\phi,N} \le C\exp[O(\|\phi\|^2)]$ (with L-dependent constant in the big-O), and this provides additional information concerning the growth in ϕ. We are not always careful to distinguish the special nature of $\|\cdot\|_N$, but inspection reveals that our conclusions indeed hold with this choice.

1.2.2 Accuracy of perturbative analysis

One of our main results is a proof of a version of (1.27) that goes beyond formal power series. The version we prove is a local one, which permits accurate estimates with errors bounded uniformly in the volume. However, the local analysis comes with a cost, which is that an explicit second-order leading term arises along with the third-order error.

For simplicity, for the present discussion we set $\lambda = q = 0$ so that observables play no role. In this setting, a particular case of what we prove is that for $b \in \mathcal{B}_j$ and $B \in \mathcal{B}_{j+1}$,

$$\tilde{I}_{\rm pt}^{B\setminus b}\mathbb{E}_{j+1}\theta I(V,b) \approx \tilde{I}_{\rm pt}^{B}\Big(1 - \frac{1}{2}\mathbb{E}_{j+1}\theta(V_j(b);V_j(\Lambda\setminus b)\Big), \tag{1.60}$$

where the *truncated expectation* (or *covariance*) is defined by

$$\mathbb{E}_C(A;B) = \mathbb{E}_C(AB) - (\mathbb{E}_C A)(\mathbb{E}_C B). \tag{1.61}$$

We prove precise versions of (1.60) with third-order error estimates, for both norm pairs (1.58)–(1.59). For example, in the proof of Proposition 2.6, for the norm pair (1.58) we show that

$$\|\tilde{I}_{\rm pt}^{B\setminus b}\mathbb{E}_{j+1}(\theta I(V,b) - \tilde{I}_{\rm pt}(b)) + \tilde{I}_{\rm pt}^{B}\frac{1}{2}\mathbb{E}_{j+1}\theta(V_j(b);V_j(\Lambda\setminus b)\|_{T_{0,j+1}(\ell_{j+1})} \le O(\tilde{g}_j^3). \tag{1.62}$$

The bound on the right-hand side is third-order as desired, but there is a second-order leading term on the left-hand side. Its origin can be seen from a small extension of the argument in [6, Section 2], as follows. Proceeding as in [6, Section 2], formally, to a third-order error, we obtain

$$\mathbb{E}\theta I(b) \approx e^{-\mathbb{E}\theta V(b)}\left[1 + \mathbb{E}\theta W(b) + \frac{1}{2}\mathbb{E}\theta(V(b);V(b))\right]. \tag{1.63}$$

The bilinear term $W(b)$ involves $V(b)$ in one argument and $V(\Lambda)$ in the other, and its partner to make the argument of [6, Section 2] apply here has to be $\frac{1}{2}\mathbb{E}\theta(V(b);V(\Lambda))$ rather than $\frac{1}{2}\mathbb{E}\theta(V(b);V(b))$. Thus we rewrite the right-hand side as

$$\mathbb{E}\theta I(b) \approx e^{-\mathbb{E}\theta V(b)}\left[1 + \mathbb{E}\theta W(b) + \frac{1}{2}\mathbb{E}\theta(V(b);V(\Lambda)) - \frac{1}{2}\mathbb{E}\theta(V(b;V(\Lambda\setminus b))\right]. \tag{1.64}$$

After multiplication by $\tilde{I}_{\rm pt}^{B\setminus b}$, the extra term produces $-\tilde{I}_{\rm pt}^{B}\frac{1}{2}\mathbb{E}\theta(V(b;V(\Lambda\setminus b))$, which is what appears in (1.62).

In Proposition 2.5, we prove that the leading term in the perturbative estimates we require is indeed second order. This is a straightforward consequence of the stability bounds. The fact that the remainder beyond the leading term is third order is proved in Proposition 2.6, which is more substantial, and is our full implementation of the formal arguments of [6, Section 2]. For the reasons discussed in Section 1.2.1, we need versions of these two propositions for both norm pairs.

1.2.3 Loc and the crucial contraction

The renormalisation group creates an infinite-dimensional dynamical system, which has a finite number of relevant or marginal directions and infinitely many irrelevant directions. A crucial

aspect of our analysis is to employ the operator Loc defined and developed in [14] to extract the relevant and marginal parts of a functional of the fields, with $(1 - \mathrm{Loc})$ projecting onto the irrelevant parts. The specific result we prove in this respect is Proposition 2.8. A special case of Proposition 2.8 is as follows.

Let X be a small set as defined above (1.37). Let U be the smallest collection of blocks in $\mathcal{B}_{j+1}$ which contains X (U is the *closure* $U = \overline{X}$). Let $F(X) \in \mathcal{N}(X^{\square})$ be such that $\mathrm{Loc}_X F = 0$; this should be interpreted as a statement that $F(X)$ is irrelevant for the renormalisation group. We prove in Proposition 2.8 that, under appropriate assumption on V,

$$\|\tilde{I}^{U\setminus X}\mathbb{E}\theta\left(\tilde{I}^{X}F(X)\right)\|_{j+1} \leq \mathrm{const} L^{-d-1}\|F(X)\|_j, \tag{1.65}$$

where the pair of norms is given by either choice of (1.58) or (1.59). The number of distinct X with closure U produces an entropic factor of order L^d, and hence

$$\sum_{X\in\mathcal{S}_j:\overline{X}=U} \|\tilde{I}^{U\setminus X}\mathbb{E}\theta\left(\tilde{I}^{X}F(X)\right)\|_{j+1} \leq \mathrm{const} L^{-1}\|F(X)\|_j. \tag{1.66}$$

Thus a contractive factor L^{-1} remains also after summation. This plays a crucial role in [15] in showing that the coordinate of the dynamical system that is meant to represent the irrelevant directions is in actual fact contractive.

1.3 Parameters and domains

In this section, we reformulate estimates on the covariance decomposition that are stated in [6], we specify the parameters that define the T_ϕ norms we use, we define the small parameters $\epsilon_V, \bar{\epsilon}$ that permeate our analysis, and we discuss the domains for V which ensure stability of I.

1.3.1 Estimate on covariance decomposition

We now discuss the size of the covariances arising in the covariance decomposition, in more detail. Recall from (1.35) the definition $\ell_j = \ell_0 L^{-j[\phi]}$. We may regard a covariance C as a test function depending on two arguments x, y, and with this identification its $\Phi_j(\ell_j)$ norm is

$$\|C\|_{\Phi_j(\ell_j)} = \ell_j^{-2} \sup_{x,y\in\Lambda} \sup_{|\alpha_1|_1+|\alpha_2|_1\leq p_\Phi} L^{(|\alpha_1|_1+|\alpha_2|_1)j}|\nabla_x^{\alpha_1}\nabla_y^{\alpha_2}C_{x,y}|, \tag{1.67}$$

where α_i is a multi-index. The norm of the covariance C_j in the covariance decomposition can be estimated using an improved version of (1.33) from [3, 6].

For this, given $\Omega > 1$ we define the Ω-*scale* j_Ω by

$$j_\Omega = \inf\{k \geq 0 : |\beta_j| \leq \Omega^{-(j-k)}\|\beta\|_\infty \text{ for all } j\}, \tag{1.68}$$

and we set

$$\chi_j = \Omega^{-(j-j_\Omega)_+}. \tag{1.69}$$

The Ω-scale indicates a scale at which the mass term in the covariance starts to play a dominant role in dramatically reducing the size of the covariance; further discussion of this point can be

found in [6, Section 6]. It is within a constant of the value j_m defined by $j_m = \lfloor \log_{L^2} m^{-2} \rfloor$, as shown in [6, Proposition 4.4], and χ_j could alternately be defined in terms of j_m. We always take the infinite volume limit before letting $m^2 \downarrow 0$, so we may assume that $m^2 \in [\varepsilon L^{-2(N-1)}, \delta^2]$ for small fixed δ.

It is shown in [6, (6.102)] that there is an L-independent constant c such that for $m^2 \in [0, \delta]$ and $j = 1, \dots, N-1$, or for $m^2 \in [\varepsilon L^{-2(N-1)}, \delta]$ for N large in the special case $C_j = C_{N,N}$,

$$|\nabla_x^\alpha \nabla_y^\beta C_{j;x,y}| \le c \chi_j L^{-(j-1)(2[\phi]+(|\alpha|_1+|\beta|_1))}. \tag{1.70}$$

Let

$$C_{j*} = \begin{cases} C_j & j < N \\ C_{N,N} & j = N. \end{cases} \tag{1.71}$$

By (1.70), given $\mathfrak{c} \in (0, 1]$ we can choose ℓ_0 large depending on L to obtain, for $j = 1, \dots, N$,

$$\|C_{j*}\|_{\Phi_j^+(\ell_j)} \le \mathfrak{c}\chi_j \le \min\{\mathfrak{c}, \chi_j\}, \tag{1.72}$$

where Φ^+ refers to the norm (1.67) with p_Φ replaced by $p_\Phi + d$.

Let $c_G = c(\alpha_G)$ be the (small) constant of [13, Proposition 3.20]. We fix the value $\mathfrak{c} = \frac{1}{10} c_G$. Then [13, Proposition 3.20] ensures that

$$\max_{k=j,j+1} \mathbb{E}_{k*}(G_j(X))^{10} \le 2^{|X|_j} \qquad X \in \mathcal{P}_j, \tag{1.73}$$

where $|X|_j$ denotes the number of scale-j blocks comprising X (the constants 10 and 2 in (1.73) are convenient but somewhat arbitrary choices). The use of Φ^+ in (1.72) is to satisfy the hypotheses of [13, Proposition 3.20].

1.3.2 Choice of norm parameters

We restrict attention here to $d = 4$.

For the G norm, for the boson and fermion fields we choose ℓ_0 according to (1.72) and set

$$\mathfrak{h}_j = \ell_j = \ell_0 L^{-j[\phi]}. \tag{1.74}$$

For the $\tilde{G}$ norm, we fix a parameter k_0 (small, chosen as discussed under Proposition 2.1), we set

$$\mathfrak{h}_j = h_j = k_0 \tilde{g}_j^{-1/4} L^{-jd/4}. \tag{1.75}$$

We assume that $\tilde{g}_j$ can be taken to be as small as desired (uniformly in j, and depending on L), and that

$$\frac{1}{2}\tilde{g}_{j+1} \le \tilde{g}_j \le 2\tilde{g}_{j+1} \tag{1.76}$$

(the above two inequalities hold for the sequence $\bar{g}_j$ by [6, (6.101)]). For the observables, we set

$$\mathfrak{h}_{\sigma,j} = \begin{cases} \tilde{g}_j L^{(j \wedge j_{ab})[\phi]} 2^{(j-j_{ab})_+} & \mathfrak{h} = \ell \\ \tilde{g}_j^{1/4} L^{(j \wedge j_{ab})[\phi]} 2^{(j-j_{ab})_+} & \mathfrak{h} = h; \end{cases} \tag{1.77}$$

see Remark 3.3 for motivation of this definition. By (1.76), the above choices obey:

$$\mathfrak{h}_j \ge \ell_j, \qquad \frac{\mathfrak{h}_{j+1}}{\mathfrak{h}_j} L^{[\phi]} \le 2, \qquad \frac{\mathfrak{h}_{\sigma,j+1}}{\mathfrak{h}_{\sigma,j}} \le \text{const} \begin{cases} L^{[\phi]} & j < j_{ab} \\ 1 & j \ge j_{ab}. \end{cases} \tag{1.78}$$

1.3.3 Definition of small parameter ϵ_V

The stability estimates are expressed in terms of domains defined via parameters ϵ_V and $\epsilon_{g\tau^2}$, which we discuss now. Given $V \in \mathcal{Q}$, we write $V_\varnothing = \sum_M M$ for the decomposition of its bulk part as a sum of individual field monomials such as $\nu\phi\bar{\phi}$, $\nu\psi\bar{\psi}$, $z(\Delta\phi)\bar{\phi}$, and so on. Then, for $0 \le j \le N$, we define

$$\epsilon_V = \epsilon_{V,j} = L^{dj} \sum_{M:\pi_* M=0} \|M_0\|_{T_{0,j}(\mathfrak{h}_j)} + 2|\lambda|\mathfrak{h}_j\mathfrak{h}_{\sigma,j} + |q|\mathfrak{h}_{\sigma,j}^2, \tag{1.79}$$

where M_0 denotes the monomial M_x evaluated at $x = 0$. Thus ϵ_V is a function (in fact, a norm) of the coupling constants in V and of the parameters $\mathfrak{h}_j$ and $\mathfrak{h}_{\sigma,j}$ which define the T_0 semi-norm. The value of ϵ_V depends on the scale j, but we often leave this implicit in the notation. It measures the size of V on a block $B \in \mathcal{B}_j$ consisting of L^{dj} points, and is worst case in the sense that it includes a contribution from observables whether or not the points a or b lie in B.

The term $g\tau^2$ plays a special role in providing the important factor $e^{-g|\phi|^4}$ in e^{-V}, and we define

$$\epsilon_{g\tau^2} = \epsilon_{g\tau^2,j} = L^{dj}\|g\tau_0^2\|_{T_{0,j}(\mathfrak{h}_j)}. \tag{1.80}$$

By definition, $\epsilon_{g\tau^2} \le \epsilon_V$. Also, it follows from [13, Proposition 3.9] that there is a universal constant $C_0 > 0$ such that

$$C_0^{-1}|g|\mathfrak{h}_j^4 L^{dj} \le \epsilon_{g\tau^2} \le C_0|g|\mathfrak{h}_j^4 L^{dj}. \tag{1.81}$$

1.3.4 Stability domains

To enable the use of analyticity methods in [15], we employ complex coupling constants. Given a (large) constant $C_{\mathcal{D}}$, we define a domain

$$\begin{aligned}\mathcal{D}_j = \{(g,\nu,z,y,\lambda,q) \in \mathbb{C}^6 : C_{\mathcal{D}}^{-1}\tilde{g}_j < \mathrm{Re}g < C_{\mathcal{D}}\tilde{g}_j,\ |\mathrm{Im}g| < \tfrac{1}{10}\mathrm{Re}g,\\ |x| \le r_x \text{ for } x \ne g\},\end{aligned} \tag{1.82}$$

where r_x is defined by

$$\begin{aligned} L^{2j}r_{\nu,j} = r_{z,j} = r_{y,j} = C_{\mathcal{D}}\tilde{g}_j, \qquad r_{\lambda,j} = C_{\mathcal{D}},\\ L^{2j_{ab}[\phi]}2^{2(j-j_{ab})}r_{q,j} = \begin{cases} 0 & j < j_{ab} \\ C_{\mathcal{D}} & j \ge j_{ab}. \end{cases}\end{aligned} \tag{1.83}$$

We also use two additional domains in $\mathbb{C}^6$, which depend on the value of $\mathfrak{h}$ (namely $\mathfrak{h} = \ell$ or $\mathfrak{h} = h$), as well as on parameters $\alpha, \alpha', \alpha'' > 0$. Given these parameters, we define

$$\bar{\mathcal{D}}_j(\ell) = \{V \in \mathcal{Q} :\ |\mathrm{Im}g| < \tfrac{1}{5}\mathrm{Re}g,\ \epsilon_{V,j}(\ell_j) \le \alpha''\tilde{g}_j\}, \tag{1.84}$$

$$\bar{\mathcal{D}}_j(h) = \{V \in \mathcal{Q} :\ |\mathrm{Im}g| < \tfrac{1}{5}\mathrm{Re}g,\ \alpha \le \epsilon_{g\tau^2,j}(h_j),\ \epsilon_{V,j}(h_j) \le \alpha'\}. \tag{1.85}$$

We permit the parameters $\alpha, \alpha' > 0$ to depend on $C_{\mathcal{D}}$, and $\alpha'' = \alpha''_L > 0$ to depend on $C_{\mathcal{D}}, L$. Their specific values are of no importance. We sometimes need versions with larger α', α'' and smaller α, and we denote these by $\bar{\mathcal{D}}'_j$. This is the case in the following proposition, which is proved in Section 3.

Proposition 1.4. *Let $d = 4$. If $V \in \mathcal{D}_j$ then there is a choice of parameters defining the domains* (1.84)–(1.85) *such that*

$$V \in \bar{\mathcal{D}}_j(\ell) \cap \bar{\mathcal{D}}_j(h) \qquad (j \le N), \tag{1.86}$$

and if $V \in \bar{\mathcal{D}}_j(\mathfrak{h})$ (for $\mathfrak{h} = h$ or ℓ) then with a new choice of parameters for $\bar{\mathcal{D}}'$,

$$V_{\mathrm{pt},j+1}(V) \in \bar{\mathcal{D}}'_j(\mathfrak{h}) \cap \bar{\mathcal{D}}'_{j+1}(\mathfrak{h}) \qquad (j < N). \tag{1.87}$$

The domains $\bar{\mathcal{D}}_j$ is the principal domain for V throughout the paper. By Proposition 1.4, we know that $\mathcal{D}_j \subset \bar{\mathcal{D}}_j(\mathfrak{h}_j)$ for both $\mathfrak{h} = \ell$ and $\mathfrak{h} = h$, so all assertions valid for $V \in \bar{\mathcal{D}}_j$ are valid for $V \in \mathcal{D}_j$. In particular, (1.86) asserts that if $V \in \mathcal{D}_j$, then

$$\alpha \le \epsilon_{g\tau^2,j}(h), \qquad \epsilon_{V,j} \le \begin{cases} \alpha''_L \tilde{g}_j & \mathfrak{h} = \ell \\ \alpha' & \mathfrak{h} = h. \end{cases} \tag{1.88}$$

From (1.81) and (1.75), we see that

$$\epsilon_{g\tau^2}(h) \asymp k_0^4, \tag{1.89}$$

where k_0 is the small constant in the definition of h_j. From (1.88) we see that the $g\tau^2$ term dominates V in the h-norm, in the sense that

$$\epsilon_V(h) \le \alpha' \alpha^{-1} \epsilon_{g\tau^2}(h). \tag{1.90}$$

Together with the lower bound on $\epsilon_{g\tau^2}(h)$, this is important in using the $e^{-g\tau^2}$ factor in e^{-V} to obtain effective stability bounds. A bound like (1.90) also holds for the case $\mathfrak{h} = \ell$, but with an L-dependent constant; this follows since $\epsilon_V(\ell)$ and $\epsilon_{g\tau^2}(\ell)$ are both of order $\tilde{g}_j$ by (1.81) and (1.86). However in this case, since we are interested in situations where $\tilde{g}_j \to 0$ as $j \to \infty$, we do not have a uniform lower bound on $\epsilon_{g\tau^2}(\ell)$.

Remark 1.5. Our analysis throughout the paper rests on the estimates of Proposition 1.4 but does not depend on the particular form of the observable terms in (1.10) and their counterparts on the right-hand side of (1.79). If different observable terms were used instead then there is no significant change in the analysis as long as the statements of Proposition 1.4 remain valid; this observation is useful in [9].

1.3.5 Definition of small parameter $\bar{\epsilon}$

An additional small parameter which is important for our analysis is $\bar{\epsilon} = \bar{\epsilon}(\mathfrak{h})$, which takes on different values for the two cases $\mathfrak{h} = \ell$ and $\mathfrak{h} = h$. Recall that the sequence $\chi_j = \Omega^{-(j-j_\Omega)_+}$ was defined in (1.69). We define

$$\bar{\epsilon} = \bar{\epsilon}_j = \begin{cases} \chi_j^{1/2} \tilde{g}_j & \mathfrak{h}_j = \ell_j \\ \chi_j^{1/2} \tilde{g}_j^{1/4} & \mathfrak{h}_j = h_j. \end{cases} \tag{1.91}$$

In view of our assumption throughout the paper that $\tilde{g}_j$ is small (uniformly in j, and small depending on L), we can assume that $\bar{\epsilon}$ is as small as desired (depending on L). The sequence χ_j occurring in $\bar{\epsilon}^2$ provides useful exponential decay beyond the Ω-scale (1.68).

The small parameter $\bar{\epsilon}$ plays a role in many aspects of the paper. For example, it arises as an upper bound for W of (1.19)–(1.20) and for P of (1.24), in the sense that there is an L-dependent constant c_L such that for $1 \le j \le N$ and $V \in \bar{\mathcal{D}}_j$,

$$\max_{B \in \mathcal{B}_j} \|W_j(V,B)\|_{T_{0,j}(\mathfrak{h}_j)} \le c_L \bar{\epsilon}^2, \tag{1.92}$$

$$\max_{B \in \mathcal{B}_j} \|P_j(V,B)\|_{T_{0,j}(\mathfrak{h}_j)} \le c_L \bar{\epsilon}^2. \tag{1.93}$$

The inequalities (1.92)–(1.93) are proved in Proposition 4.1 below.

2 Main results

We now state our main results. We begin in Section 2.1 with stability estimates on the interaction I and a statement of the analyticity of I in the polynomial V. In Section 2.2 we state our results concerning the accuracy of the perturbative calculations of [6]. Finally, in Section 2.3, we state estimates on Gaussian expectation, and on the operator $(1 - \mathrm{Loc})$ which extracts the irrelevant part of an element of $\mathcal{N}$; both of these estimates involve advancement of the scale. Proofs are deferred to Sections 5–7.

2.1 Stability estimates

In this section, we state stability estimates on I, and formulate the analyticity of I in V. Proofs are given in Section 5.

Fundamental stability bounds are given in the following proposition, which is valid for *arbitrary* choice of $\mathfrak{h}$ in the definition of the norms, with corresponding $\epsilon_V, \epsilon_{g\tau^2}$ as defined in Section 1.3.3. According to (1.92), if $V \in \bar{\mathcal{D}}$ then $\|W(V,B)\|_{T_0}$ (which occurs in the hypothesis) is of order $\bar{\epsilon}^2$ so can be made as small by the requirement that $\tilde{g}_j$ be uniformly sufficiently small. Recall that the norm $\|\phi\|_{\tilde{\Phi}(X)}$ was defined in (1.40); it appears in the last exponent in (2.2). All norms in Proposition 2.1 are at scale j. The proof of (2.2) makes use of the Sobolev inequality proved in Appendix A to take advantage of the quartic decay in $e^{-g\tau^2}$. The restriction to $j < N$ in (2.2) is connected with the fact that we do not define the $\tilde{G}$ norm at scale N.

Proposition 2.1. *Let $V \in \mathcal{Q}$ with $0 \le |\mathrm{Im} g| \le \frac{1}{2}\mathrm{Re} g$. Let $j \le N$ and $B \in \mathcal{B}_j$. Let $\omega = \max_{B \in \mathcal{B}_j} \|W(V,B)\|_{T_0}$ and fix any $u \ge 6(L^{2d}\omega)^{1/3}$. Let $F \in \mathcal{N}(B^\square)$ be a polynomial of degree $r \le p_{\mathcal{N}}$. Let I^* denote any one of the following choices:*
(a) $I_j(B)$, (b) $\tilde{I}_j(B)$, (c) $\tilde{I}_j(B \setminus X)$ with $X \in \mathcal{S}_{j-1}(B)$, (d) any of (a-c) with any number of their $1+W$ factors omitted (thus, in particular, including the case $\mathcal{I}(B)$ of (1.21)).
(i) Then

$$\|I^* F\|_{T_\phi} \le \left(\frac{2r}{u}\right)^r \|F\|_{T_0} e^{O(\epsilon_V + u)(1 + \|\phi\|^2_{\Phi(B^\square)})}. \tag{2.1}$$

(ii) Suppose in addition that there is a constant C such that $\epsilon_V \le C\epsilon_{g\tau^2}$. Fix any $q \ge 0$, and let $q_1 = q + 2u\epsilon^{-1}_{g\tau^2}$. Then for $j < N$,

$$\|I^* F\|_{T_\phi} \le \left(\frac{2r}{u}\right)^r \|F\|_{T_0} e^{O[(1+q_1^2)\epsilon_{g\tau^2} + u]} e^{-q\epsilon_{g\tau^2}\|\phi\|^2_{\Phi(B^\square)}} e^{O(1+q_1)\epsilon_{g\tau^2}\|\phi\|^2_{\tilde{\Phi}(B^\square)}}. \tag{2.2}$$

When $r = 0$, (2.1)–(2.2) both hold with the prefactor $\left(\frac{2r}{u}\right)^r$ replaced by 1.

Notation. We write $a \prec b$ when there is a constant $c > 0$, independent of L and j, such that $a \le cb$. If there is an L-dependent such constant, we write $a \prec_L b$. We write $a \asymp b$ when $a \prec b$ and $b \prec a$.

We now discuss applications of Proposition 2.1 under the assumption that $V \in \bar{\mathcal{D}}_j$ of (1.84)–(1.85).

Application of (2.1). We apply (2.1) with the choice $\mathfrak{h}_j = \ell_j$ of (1.74). By Proposition 1.4 and (1.92), with this choice ϵ_V and $\|W(B)\|_{T_0}$ can be made as small as desired when $V \in \bar{\mathcal{D}}(\ell_j)$ with sufficiently small $\tilde{g}_j$ (small uniformly in j). Hence u can also be chosen small enough that $O(\epsilon_V + u) \le \frac{1}{2}$ (say), but with u bounded away from zero uniformly in scale j and in V so that the factor u^{-r} remains bounded. With these choices, and with the fluctuation-field regulator defined by (1.38), we can restate (2.1) for $V \in \bar{\mathcal{D}}_j$ as

$$\|I^*F\|_{T_\phi(\ell)} \le \left(\frac{2r}{u}\right)^r \|F\|_{T_0(\ell)}\, 2e^{\|\phi\|_\Phi^2} = \left(\frac{2r}{u}\right)^r \|F\|_{T_0(\ell)}\, 2G(B,\phi), \tag{2.3}$$

again with the convention that $(\frac{2r}{u})^r = 1$ when $r = 0$.

Application of (2.2). We apply (2.1) with the choice $\mathfrak{h}_j = h_j$ of (1.75). By (1.89) and (1.92), for $V \in \bar{\mathcal{D}}_j(h_j)$, with this choice

$$\epsilon_{g\tau^2} \asymp k_0^4, \qquad \|W(B)\|_{T_0} \prec_L \tilde{g}_j^{1/2}. \tag{2.4}$$

We choose k_0 small but bounded away from 0 uniformly in V and j, and take $u = \epsilon_{g\tau^2}$. Then

$$(1+q_1^2)\epsilon_{g\tau^2} + u = \left(2 + (q+2)^2\right)\epsilon_{g\tau^2}, \qquad (1+q_1)\epsilon_{g\tau^2} = (3+q)\epsilon_{g\tau^2}. \tag{2.5}$$

We conclude from (2.2) that, for $V \in \bar{\mathcal{D}}(h_j)$,

$$\|I^*F\|_{T_\phi(h)} \le \left(\frac{2r}{ak_0^4}\right)^r \|F\|_{T_0(h)}\, 2e^{-q\epsilon_{g\tau^2}\|\phi\|^2_{\Phi(B^\square,h)}} e^{(3+q)\epsilon_{g\tau^2}\|\phi\|^2_{\tilde{\Phi}(B^\square,h)}}, \tag{2.6}$$

with the usual convention when $r = 0$. Since $h \ge \ell$, we have $\|\phi\|_{\Phi(\mathfrak{h})} \le \|\phi\|_{\Phi(\ell)}$ and hence also $\|\phi\|_{\tilde{\Phi}(\mathfrak{h})} \le \|\phi\|_{\tilde{\Phi}(\ell)}$. This allows us to conclude from (2.6) that, for $V \in \bar{\mathcal{D}}(h_j)$, if $q \le \bar{q}$ for some fixed $\bar{q} > 0$ then we can choose k_0 small depending on $\bar{q}$ and γ such that

$$\|I^*F\|_{T_\phi(h)} \le \left(\frac{2r}{ak_0^4}\right)^r \|F\|_{T_0(h)}\, 2e^{-qak_0^4\|\phi\|^2_{\Phi(B^\square,h)}} \tilde{G}^\gamma(B,\phi). \tag{2.7}$$

Vanishing at weighted infinity. In (2.3), a stronger bound in which $G(B,\phi)$ is replaced by a smaller power $G^\gamma(B,\phi)$ also holds, by the same proof. In combination with (2.7), and with $\mathcal{G}$ denoting G when $\mathfrak{h} = \ell$ and $\tilde{G}$ when $\mathfrak{h} = h$, in either case this shows that if $V \in \bar{\mathcal{D}}$ then

$$\lim_{\|\phi\|_{\Phi(B^\square)} \to \infty} \|I^*F\|_{T_\phi} \mathcal{G}(X,\phi)^{-\gamma} = 0. \tag{2.8}$$

This fact is useful in [15] to establish the property used there called "vanishing at weighted infinity."

The following proposition extends and reformulates the above estimates in terms of the four norms $\|\cdot\|_j, \|\cdot\|_{j+1}$ appearing in *either* of (1.58)–(1.59). However, here and throughout the paper, as discussed in Remark 1.3, statements about the scale-N norm are to be interpreted as applying *only* to the G^{10}_{N-1} norm, and not also to the $\tilde{G}$ norm: scale-N is always considered to correspond to $j+1$ and never to j in (1.58)–(1.59).

Proposition 2.2. *Let I_* denote either of $I_j, \tilde{I}_{\rm pt}$, with $j_* = j$ for I_j, and* either *$j_* = j$ or $j_* = j+1$ for $\tilde{I}_{\rm pt}$. We assume $j_* \le N$. Alternately, let I_* denote any of the above with any number of their $1+W$ factors omitted. Let $B \in \mathcal{B}_j$. Let $V \in \bar{\mathcal{D}}_j$ and let $F \in \mathcal{N}(B^\square)$ be a gauge-invariant polynomial in the fields of degree at most $p_{\mathcal{N}}$ with $\pi_{ab} F = 0$ if $j < j_{ab}$. Then*

$$\|I_*(B)F\|_{j_*} \prec \|F\|_{T_{0,j}}, \tag{2.9}$$

$$\|I_*(B)\|_{j_*} \le 2, \tag{2.10}$$

$$\|I_*^{-B}\|_{T_{0,j_*}} \le 2. \tag{2.11}$$

In addition, for $j+1 \le N$ and for a scale-$(j+1)$ block $\hat{B} \in \mathcal{B}_{j+1}$, and for X either a small set $X \in \mathcal{S}_j$ or the empty set $X = \varnothing$,

$$\|\tilde{I}_{\rm pt}^{\hat{B}\setminus X}\|_{j+1} \le 2. \tag{2.12}$$

The following proposition states our analyticity result for the interaction, again in terms of the four norms $\|\cdot\|_j, \|\cdot\|_{j+1}$ appearing in (1.58)–(1.59). We show that I is analytic in V by proving that there is a norm-convergent expansion of I in powers of V.

Proposition 2.3. *Let I_* denote either of $I, \tilde{I}_{\rm pt}$, with $j_* = j$ for I, and* either *$j_* = j$ or $j_* = j+1$ for $\tilde{I}_{\rm pt}$. We assume $j_* \le N$. Alternately, let I_* denote any of the above with any number of their $1+W$ factors omitted. Let $B \in \mathcal{B}_j$. Then $I(B)$ and $\tilde{I}_{\rm pt}(B)$ are analytic functions of $V \in \bar{\mathcal{D}}_j$, taking values in $\mathcal{N}(B^\square), \|\cdot\|_{j_*}$. In addition, $I(B)^{-1}$ is an analytic function of $V \in \bar{\mathcal{D}}_j$ taking values in $\mathcal{N}(B^\square), \|\cdot\|_{T_{0,j}}$.*

Recall that $\bar{\epsilon}$ was defined in (1.91), and that we use $\mathfrak{h} = \ell$ for quantities related to the norm pair (1.58), and $\mathfrak{h} = h$ for the norm pair (1.59). The following proposition measures the effect of a change in I due to a change in V that is appropriately bounded by $\bar{\epsilon}$.

Proposition 2.4. *Let $j < N$, $B \in \mathcal{B}$, $V \in \bar{\mathcal{D}}$, $Q \in \mathcal{Q}$ with $\|Q(B)\|_{T_0} \prec \bar{\epsilon}$, and set $\hat{I} = I(V-Q)$ and $I = I(V)$. Then $V - Q \in \bar{\mathcal{D}}'$, $\hat{I}(B)$ obeys the I_* estimates of Proposition 2.2, is an analytic function of $V \in \bar{\mathcal{D}}$ taking values in $\mathcal{N}(B^\square), \|\cdot\|_j$, and obeys the estimates*

$$\|\hat{I}(B) - I(B)\|_j \prec \bar{\epsilon}, \tag{2.13}$$

$$\|\hat{I}(B) - I(B)(1+Q(B))\|_{T_0} \prec_L \bar{\epsilon}^2. \tag{2.14}$$

All quantities and norms are at scale j, norms are computed with either $\mathfrak{h} = \ell$ or $\mathfrak{h} = h$, and (2.13) *holds for either choice of $\|\cdot\|_j$ in* (1.58)–(1.59).

2.2 Perturbative interaction estimates

In this section, we formulate two propositions which enable a rigorous implementation of the formal perturbative calculations of [6, Section 2]. The two propositions are applied in [15, Section 5.2]. Their statements are in terms of the small parameter $\bar{\epsilon}$ defined in (1.91).

Recall the map θ defined below (1.14), the polynomial V_{pt} defined in (1.25) (and above (1.29) for scale-N), and $\tilde{I}_{\mathrm{pt}}$ defined in (1.28). For $B \in \mathcal{B}_j$ and $X \in \mathcal{P}_j$, we define $\delta I^X \in \mathcal{N}(\mathbf{\Lambda} \sqcup \mathbf{\Lambda}')$ by

$$\delta I(B) = \theta I_j(B) - \tilde{I}_{\mathrm{pt}}(B) = \theta I_j(V, B) - \tilde{I}_{j+1}(V_{\mathrm{pt}}, B), \qquad \delta I^X = \prod_{B \in \mathcal{B}_j(X)} \delta I(B). \tag{2.15}$$

For small sets $U \in \mathcal{S}_{j+1}$ we define

$$h_{\mathrm{red}}(U) = \sum_{X \in \overline{\mathcal{P}}_j(U) : |X|_j \le 2} \tilde{I}_{\mathrm{pt}}^{-X} \mathbb{E}_{j+1} \delta I^X, \tag{2.16}$$

where $|X|_j$ denotes the number of scale-j blocks in X, and $X \in \overline{\mathcal{P}}_j(U)$ indicates the restriction that U is the smallest polymer in $\mathcal{P}_{j+1}$ that contains X. The subscript "red" indicates that h is "reduced" by the restriction $|X|_j \le 2$. (In [15, (5.10)] we define a version without this restriction.)

For [15], we need to compute h_{red} accurately to second order in $\bar{\epsilon}$. For this, we first recall from (1.61) the definition of the truncated expectation

$$\mathbb{E}_C(A; B) = \mathbb{E}_C(AB) - (\mathbb{E}_C A)(\mathbb{E}_C B). \tag{2.17}$$

We also define (cf. (1.18))

$$\mathbb{E}_{\pi, C}(A; B) = \mathbb{E}_C(A; \pi_\varnothing B) + \mathbb{E}_C(\pi_* A; B). \tag{2.18}$$

Then for $(U, B) \in \mathcal{S}_{j+1} \times \mathcal{B}_{j+1}$ we define $h_{\mathrm{lead}}(U, B)$ by

$$h_{\mathrm{lead}}(U, B) = \begin{cases} -\frac{1}{2} \mathbb{E}_{\pi, j+1} \theta(V_j(B); V_j(\Lambda \setminus B)) & U = B \\ \frac{1}{2} \mathbb{E}_{\pi, j+1} \theta(V_j(B); V_j(U \setminus B)) & U \supset B, |U|_{j+1} = 2 \\ 0 & \text{otherwise,} \end{cases} \tag{2.19}$$

where we have abbreviated the subscript C_{j+1} to $j+1$ on $\mathbb{E}$. For $U \in \mathcal{P}_{j+1}$ we define

$$h_{\mathrm{lead}}(U) = \sum_{B \in \mathcal{B}_{j+1}(U)} h_{\mathrm{lead}}(U, B). \tag{2.20}$$

Due to the finite-range property (1.2),

$$\sum_{U \supset B : U \neq B} h_{\mathrm{lead}}(U, B) = \frac{1}{2} \mathbb{E}_{\pi, j+1} \theta(V_j(B); V_j(\Lambda \setminus B)), \tag{2.21}$$

and therefore h_{lead} obeys the identity

$$\sum_{U \supset B} h_{\mathrm{lead}}(U, B) = 0. \tag{2.22}$$

The following two propositions, which are proved in Section 6, show that h_{lead} is second order in $\bar{\epsilon}$, and that $h_{\mathrm{lead}}(U)$ is the leading part of $h_{\mathrm{red}}(U)$. The latter is a much more substantial result than the former, and is our implementation of the formal power series statement of [6, Proposition 2.1].

Proposition 2.5. *There is a positive constant* $c_{\text{lead}} = c_{\text{lead}}(L)$ *such that for* $j < N$, $V \in \bar{\mathcal{D}}_j$ *and* $(U, B) \in \mathcal{S}_{j+1} \times \mathcal{B}_{j+1}$,

$$\|\tilde{I}_{\text{pt}}(U) h_{\text{lead}}(U, B)\|_{j+1} \le c_{\text{lead}} \bar{\epsilon}^2, \tag{2.23}$$

where $\|\cdot\|_{j+1}$ *represents either of the two options* (1.58)–(1.59)*, with corresponding* $\bar{\epsilon}$ *of* (1.91)*.*

Proposition 2.6. *There is a positive constant* $c_{\text{pt}} = c_{\text{pt}}(L)$ *such that for* $j < N$, $V \in \bar{\mathcal{D}}_j$ *and* $U \in \mathcal{S}_{j+1}$ *with* $|U|_{j+1} \in \{1, 2\}$,

$$\|\tilde{I}_{\text{pt}}(U)[h_{\text{red}}(U) - h_{\text{lead}}(U)]\|_{j+1} \le c_{\text{pt}} \bar{\epsilon}^3, \tag{2.24}$$

where $\|\cdot\|_{j+1}$ *represents either of the two options* (1.58)–(1.59)*, with corresponding* $\bar{\epsilon}$ *of* (1.91)*.*

2.3 Bound on expectation and crucial contraction

The next two propositions play a key role in our analysis of a single renormalisation group step in [15, Section 5.2].

Proposition 2.7. *There is an* $\alpha_{\mathbb{E}} > 0$ *(independent of* L*) and a* $C_{\delta V} > 0$ *(depending on* L*) such that for* $j < N$, $V \in \bar{\mathcal{D}}_j$*, disjoint* $X, Y \in \mathcal{P}_j$*, and for* $F(Y) \in \mathcal{N}(Y^{\square})$*,*

$$\|\mathbb{E}_{j+1} \delta I^X \theta F(Y)\|_{j+1} \le \alpha_{\mathbb{E}}^{|X|_j + |Y|_j} (C_{\delta V} \bar{\epsilon})^{|X|_j} \|F(Y)\|_j, \tag{2.25}$$

where the pair of norms is given by either of (1.58) *or* (1.59) *with corresponding* $\bar{\epsilon}$ *of* (1.91)*.*

The proof of Proposition 2.7 is given in Section 7.1. We refer to the important inequality (2.25) as the *integration property*. It shows that when estimating the scale-$(j+1)$ norm of an expectation of a product involving factors of $\delta I(B)$ for scale-j blocks, each factor gives rise to a small factor $\bar{\epsilon}$.

In the next proposition, the notation $U = \overline{X}$ again indicates the restriction that U is the smallest polymer in $\mathcal{P}_{j+1}$ that contains X. As in [14, Definition 1.17], we use the notation $X(\varnothing) = X$, $X(a) = X \cap \{a\}$, $X(b) = X \cap \{b\}$, and $X(ab) = X \cap \{a, b\}$. Given $X \subset \Lambda$, we define

$$\gamma(X) = L^{-d-1} + L^{-1} \mathbb{1}_{X \cap \{a,b\} \neq \varnothing}. \tag{2.26}$$

Proposition 2.8. *Let* $j < N$ *and* $V \in \mathcal{D}_j$*. Let* $X \in \mathcal{S}_j$ *and* $U = \overline{X}$*. Let* $F(X) \in \mathcal{N}(X^{\square})$ *be such that* $\pi_\alpha F = 0$ *when* $X(\alpha) = \varnothing$*. Then*

$$\|\tilde{I}_{\text{pt}}^{U \setminus X} \mathbb{E}_{C_{j+1}} \theta F(X)\|_{j+1} \prec \gamma(X) \kappa_F + \kappa_{\text{Loc}F}, \tag{2.27}$$

with $\kappa_F = \|F(X)\|_j$ *and* $\kappa_{\text{Loc}F} = \|\tilde{I}_{\text{pt}}^X \text{Loc}_X \tilde{I}_{\text{pt}}^{-X} F(X)\|_j$*, and where the pair of norms is given by either of* (1.58) *or* (1.59)*.*

The proof of the Proposition 2.8 is given in Section 7.2. We refer to the inequality (2.27) as the *crucial contraction*; its importance is discussed in Section 1.2.3 above.

3 Estimates on small parameters

In this section, we provide estimates on the small parameters $\epsilon_V, \bar{\epsilon}$ which drive our analysis. In particular, we prove Proposition 1.4.

3.1 Preliminaries

We begin with two general lemmas. The first relates $\epsilon_{V,j}$ to $\|V(B)\|_{T_{0,j}}$ for a scale-j block $B \in \mathcal{B}_j$, and the second expresses an important monotonicity property of the T_ϕ semi-norm under change of scale. Recall from [14, (1.61)] that it follows from the definition of the T_ϕ semi-norm that under the direct sum decomposition of $F \in \mathcal{N}$ due to (1.6),

$$\|F\|_{T_\phi} = \sum_{\alpha \in \varnothing, a, b, ab} \|\pi_\alpha F\|_{T_\phi} = \|F_\varnothing\|_{T_\phi} + \mathfrak{h}_\sigma \|F_a\|_{T_\phi} + \mathfrak{h}_\sigma \|F_b\|_{T_\phi} + \mathfrak{h}_\sigma^2 \|F_{ab}\|_{T_\phi}. \tag{3.1}$$

3.1.1 The T_0 semi-norm and ϵ_V

Lemma 3.1. *For $V \in \mathcal{Q}$ and $j < N$, $\epsilon_{V,j} \asymp \max_{B \in \mathcal{B}_j} \|V(B)\|_{T_{0,j}}$.*

Proof. Given $V \in \mathcal{Q}$, as in (1.79) we write $V_\varnothing = \sum_M M$ for the decomposition of its bulk part as a sum of individual field monomials such as $\nu\phi\bar{\phi}$, $\nu\psi\bar{\psi}$, $z(\Delta\phi)\bar{\phi}$, and so on. For $0 \le j \le N$, in (1.79) we defined

$$\epsilon_V = L^{dj} \sum_{M : \pi_* M = 0} \|M_0\|_{T_{0,j}(\mathfrak{h}_j)} + 2|\lambda|\mathfrak{h}_j \mathfrak{h}_{\sigma,j} + |q|\mathfrak{h}_{\sigma,j}^2. \tag{3.2}$$

By direct calculation, $\|\lambda\bar{\phi}_x\|_{T_0} = \mathbb{1}_{x=a}|\lambda|\mathfrak{h}$, $\|\lambda\phi_x\|_{T_0} = \mathbb{1}_{x=b}|\lambda|\mathfrak{h}$, and also $\|q_x\|_{T_0} = \frac{1}{2}(\mathbb{1}_{x=a} + \mathbb{1}_{x=b})|q|$. Thus the last two terms on the right-hand side of (3.2) are comparable to $\max_{B \in \mathcal{B}_j} \|\pi_* V(B)\|_{T_0}$, and it suffices to consider the case $V = \pi_\varnothing V$, which we assume henceforth.

It follows from the triangle inequality that $\|V(B)\|_{T_0} \prec \epsilon_V$, and it suffices to prove the reverse inequality. Let M be a scalar multiple of one of $g\phi\bar{\phi}\phi\bar{\phi}, g\phi\bar{\phi}\psi\bar{\psi}, \nu\phi\bar{\phi}, \ldots$. It suffices to prove that

$$\|M_0\|_{T_0}|B| \prec \|V(B)\|_{T_0}. \tag{3.3}$$

For this, we employ the pairing of [13, Definition 3.3], and seek dual test functions for the monomials. In more detail, given a monomial M we seek a unit Φ-norm test function f_M such that, for all $x \in B$, $\langle M_x, f_M \rangle = \|M_x\|_{T_0}$ but $\langle M'_x, f_M \rangle = 0$ if $M' \neq M$. It then follows that

$$\|M_0\|_{T_0} = |\langle M_x, f_M \rangle_0| = \frac{1}{|B|}|\langle M(B), f_M \rangle_0| = \frac{1}{|B|}|\langle V(B), f_M \rangle_0| \le \frac{1}{|B|}\|V(B)\|_{T_0}, \tag{3.4}$$

which is equivalent to the desired estimate for this monomial.

For the existence of f_M, we proceed as follows (cf. [14, Lemma 3.1] for related ideas). Consider first the case $M = g\phi\bar{\phi}\phi\bar{\phi}$. We choose f_M to be zero on all sequences except those of length four whose components are in the $\phi, \bar{\phi}, \phi, \bar{\phi}$ sheets, and choose it to be constant on the set of these sequences, with the constant such that f_M has unit norm. This choice can be seen to have the desired properties, and it generalises in a straightforward way to all the monomials arising from $g\tau^2$ and $\nu\tau$.

Next, we consider $M = \frac{1}{2}\sum_{e \in \mathcal{U}} (\nabla^e \phi)(\nabla^e \bar{\phi})$ (the coupling constant plays an insignificant role so we omit it for simplicity). By translation invariance, we may assume that B is centred at $0 \in \Lambda$, and since $j < N$ we can identify B with a subset of $\mathbb{Z}^d$. Let $v_{x_1,x_2} = x_1 \cdot x_2 + c$ for x_1 in the ϕ sheet and x_2 in the $\bar{\phi}$ sheet. Let $M' = \phi\bar{\phi}$. Then the pairing of v with any monomial other than M, M' vanishes. In particular, $\langle M_x, v \rangle_0 = \frac{1}{2}\sum_{e \in \mathcal{U}} \nabla^e_{x_1} \nabla^e_{x_2} v_{x,x} = d$. Also, $\langle M'_x, v \rangle_0 \asymp x \cdot x + c$, and by choosing $c \asymp L^{2j}$ such that $\sum_{x \in B}(x \cdot x + c) = 0$, we can arrange that $\langle M'(B), v \rangle_0 = 0$. Let

$f = v/\|v\|_\Phi$. Then we have $\langle V(B), f\rangle_0 = \langle M(B), f\rangle_0 = |B|\langle M_0, f\rangle_0$ and we obtain (3.3) in this case, as in (3.4).

The case $M = \bar\phi\Delta\phi$ is similar, with the test function constructed from $v_{x_1,x_2} = x_1 \cdot x_1 + c$. This completes the proof. ∎

3.1.2 Scale monotonicity

We now prove a monotonicity property of the T_ϕ semi-norm under change of scale, which is used repeatedly throughout the paper. The property is more general than our specific application, and we formulate it under assumptions on $\mathfrak{h} = (\mathfrak{h}_\phi, \mathfrak{h}_\sigma)$ obeyed by our particular choices. In our application, (3.5) with $\mathfrak{h}' = \mathfrak{h}$ follows from the last two bounds of (1.78).

Lemma 3.2. *Suppose that $F \in \mathcal{N}$ is gauge invariant and such that $\pi_{ab}F = 0$ when $j < j_{ab}$, that $\mathfrak{h}''_{\phi,j} \le \mathfrak{h}'_{\phi,j} \prec L^{-[\phi]}\mathfrak{h}_{\phi,j-1}$, that $\mathfrak{h}''_{\sigma,j} \prec \mathfrak{h}'_{\sigma,j}$, and that for all j,*

$$\mathfrak{h}'_{\sigma,j} \prec \begin{cases} L^{[\phi]}\mathfrak{h}_{\sigma,j-1} & j < j_{ab} \\ \mathfrak{h}_{\sigma,j-1} & j \ge j_{ab}, \end{cases} \qquad \mathfrak{h}'_{\sigma,j+1}\mathfrak{h}'_{\phi,j+1} \prec \mathfrak{h}_{\sigma,j}\mathfrak{h}_{\phi,j}. \tag{3.5}$$

Then

$$\|F\|_{T_{\phi,j}(\mathfrak{h}''_j)} \prec \|F\|_{T_{\phi,j}(\mathfrak{h}'_j)} \prec \|F\|_{T_{\phi,j-1}(\mathfrak{h}_{j-1})}. \tag{3.6}$$

Proof. Recall (3.1). Each $\|F_\alpha\|_{T_\phi}$ individually obeys (3.6), as a direct consequence of the definition $\|F\|_{T_\phi} = \sup_{\|g\|_\Phi \le 1} |\langle F, g\rangle_\phi|$ of the T_ϕ semi-norm in [13, Definition 3.3] and the fact that for a test function g with none of its variables corresponding to observable sheets,

$$\|g\|_{\Phi_{j-1}(\mathfrak{h}_{j-1})} \prec \|g\|_{\Phi_j(\mathfrak{h}'_j)} \le \|g\|_{\Phi_j(\mathfrak{h}''_j)}. \tag{3.7}$$

Thus we need to examine the effect of the $\mathfrak{h}_\sigma$ contributions. The first inequality of (3.6) follows immediately from the assumption that $\mathfrak{h}''_{\sigma,j} \prec \mathfrak{h}'_{\sigma,j}$.

For the second inequality, by hypothesis $\pi_{ab}F = 0$ for $j < j_{ab}$, so for $j < j_{ab}$ this component plays no role. For $j \ge j_{ab}$, we have $\mathfrak{h}'_{\sigma,j} \prec \mathfrak{h}_{\sigma,j-1}$, and we conclude that $\mathfrak{h}'^2_{\sigma,j}\|F_{ab}\|_{T_{\phi,j}(\mathfrak{h}'_j)} \prec \mathfrak{h}^2_{\sigma,j-1}\|F_{ab}\|_{T_{\phi,j-1}(\mathfrak{h}_{j-1})}$.

This leaves $\pi_\alpha F$ for $\alpha \in \{a, b\}$. These are similar, and we consider only $\alpha = a$. The fact that $\pi_a F$ is gauge invariant implies that its pairing with a test function g is zero unless g contains a σ variable and also contains a ϕ or ψ argument. Then we apply the second inequality of (3.5). This completes the proof of (3.6). ∎

3.2 The small parameter ϵ_V: Proof of Proposition 1.4

Proof of Proposition 1.4. It suffices to prove that:
(i) For $j \le N$ and $V \in \mathcal{D}_j$, there exist $a, A > 0$ (depending on $C_\mathcal{D}$) and $A_L > 0$ (depending on $C_\mathcal{D}, L$) such that

$$|\mathrm{Im} g| < \tfrac{1}{5}\mathrm{Re} g, \qquad ak_0^4 \le \epsilon_{g\tau^2,j}(h_j), \qquad \epsilon_{V,j} \le \begin{cases} A_L\tilde g_j & \mathfrak{h} = \ell \\ Ak_0 & \mathfrak{h} = h. \end{cases} \tag{3.8}$$

(ii) For $j < N$ and $V \in \bar{\mathcal{D}}_j$, the bounds (3.8) hold (with different constants) when V is replaced by $V_{\mathrm{pt},j+1}$ (and g by g_{pt}), and also when j is replaced by $j+1$.

We prove the above two statements in sequence.

(i) For $j \le N$ and $V \in \mathcal{D}_j$, the coupling constants obey

$$C_{\mathcal{D}}^{-1}\tilde{g}_j < \mathrm{Re} g < C_{\mathcal{D}}\tilde{g}_j, \quad |\mathrm{Im} g| < \tfrac{1}{10}\mathrm{Re} g < \tfrac{1}{5}\tilde{g}_j, \tag{3.9}$$

$$L^{2j}|\nu|, |z|, |y| \le C_{\mathcal{D}}\tilde{g}_j, \quad |\lambda| \le C_{\mathcal{D}}, \quad L^{2j_{ab}[\phi]}2^{2(j-j_{ab})}|q| \le \begin{cases} 0 & j < j_{ab} \\ C_{\mathcal{D}} & j \ge j_{ab}. \end{cases} \tag{3.10}$$

The first inequality of (3.8) holds by definition. It follows from [13, Proposition 3.9] that

$$\epsilon_{g\tau^2,j}(\mathfrak{h}_j) \asymp L^{dj}|g|\mathfrak{h}_j^4. \tag{3.11}$$

In particular, since $|g| \asymp \tilde{g}_j$ by hypothesis,

$$\epsilon_{g\tau^2,j}(h_j) \asymp L^{dj}|g|h_j^4 \asymp k_0^4, \tag{3.12}$$

which proves the second bound of (3.8). The last bound of (3.8) for the bulk part $V_\varnothing$ of V similarly follows from direct calculation as in [13, Proposition 3.9]; e.g., $\|\phi_x\bar{\phi}_x\|_{T_{0,i}} = \mathfrak{h}_i^2$, $\|\phi_x\bar{\phi}_x\phi_x\bar{\phi}_x\|_{T_{0,i}} = \mathfrak{h}_i^4$, $\|\phi_x\Delta\bar{\phi}_x\|_{T_{0,i}} = L^{-2i}\mathfrak{h}_i^2$, while the observables contribute

$$|\lambda|\mathfrak{h}_i\mathfrak{h}_{\sigma,i} = |\lambda| \times \begin{cases} \tilde{g}_i\ell_0(2/L)^{(i-j_{ab})_+} & \mathfrak{h} = \ell \\ k_0(2/L)^{(i-j_{ab})_+} & \mathfrak{h} = h, \end{cases} \tag{3.13}$$

$$|q|\mathfrak{h}_{\sigma,i}^2 \prec \begin{cases} \tilde{g}_i^2 & \mathfrak{h} = \ell \\ \tilde{g}_i^{1/2} & \mathfrak{h} = h \end{cases} \tag{3.14}$$

(for (3.14) we can restrict to $i \ge j_{ab}$ since otherwise $q = 0$). The combination of these bounds completes the proof of (3.8), after taking into account that ℓ_0 depends on L and $k_0^4 \le k_0$.

(ii) Let $V \in \bar{\mathcal{D}}_j$. We first consider the case $j+1 < N$, $n = j$ of (1.87). By (1.26), $V_{\mathrm{pt}} = V + 2gC_{0,0}\tau - P$, with $C = C_{j+1}$ and $P = P_j$. By (1.70),

$$\|2gC_{0,0}\tau_x\|_{T_{0,j}} \prec |g|L^{-2j}\mathfrak{h}_j^2. \tag{3.15}$$

By (1.79), Lemma 3.1, and (1.93),

$$\begin{aligned} \epsilon_{V_{\mathrm{pt}}} &\prec \epsilon_V + |g|L^{2j}\mathfrak{h}_j^2 + \epsilon_P \prec \epsilon_V + |g|L^{2j}\mathfrak{h}_j^2 + \max_{B\in\mathcal{B}_j}\|P(B)\|_{T_{0,j}} \\ &\prec \epsilon_V + |g|L^{2j}\mathfrak{h}_j^2 + O_L(\bar{\epsilon}^2). \end{aligned} \tag{3.16}$$

With the definition of $\mathfrak{h}_j$ in (1.74)–(1.75), this shows that $\epsilon_{V_{\mathrm{pt}}}$ obeys the last bound of (3.8). For the second bound of (3.8), we restrict to $\mathfrak{h} = h$, and note that the lower bound follows from the lower bound on the τ^2 term of V, together with the fact that the contribution to τ^2 from P is bounded above by $\epsilon_P \prec_L \tilde{g}_j^{1/2}$. Finally, for the bound on the imaginary part of g_{pt} we use the fact

that it changes insignificantly from the imaginary part of g, since the coupling constant g_P of P obeys $|g_P| \prec \epsilon_{g_p\tau^2,j}(\ell_j) \le \epsilon_P(\ell_j) \prec_L \tilde{g}_j^2$ (the first of these inequalities follows from (3.11)).

For the case $j+1 = N$, $n = j$ of (1.87), we simply observe that our definition of $V_{\mathrm{pt},N}$ is identical to what it would be on a torus of scale larger than N, so the bound in this case follows from the above argument applied to the torus of scale $N+1$.

For the $n = j+1$ case of (1.87), note that the computations in the proof of (i) lead to the same conclusion when $\mathfrak{h}_j$ is replaced by $\mathfrak{h}_{j+1}$ and $|B| = L^{dj}$ is replaced by $L^{d(j+1)}$, and since $\tilde{g}_{j+1} \asymp \tilde{g}_j$ by (1.76), we conclude that $V \in \bar{\mathcal{D}}_{j+1}(\ell_{j+1}) \cap \bar{\mathcal{D}}_{j+1}(h_j)$ (with adjusted constants). The desired result then follows exactly as in the proof of (ii), now with (1.93) applied at scale-$(j+1)$. This completes the proof. ∎

Remark 3.3. The choice of $\mathfrak{h}_\sigma$ in (1.77) can be motivated as follows; we discuss this for the case $\mathfrak{h} = \ell$. As a first attempt, it would be natural to choose ℓ_σ as large as possible to make the norm of $\lambda\sigma\bar{\phi}$ agree with (or be bounded by) that of $z\tau_\Delta$ on a block, namely $\tilde{g}_j L^{-2j}\ell_j^2 L^{dj} = \tilde{g}_j\ell_0^2$. The coupling constant λ is $O(1)$. The T_0 norm of $\sigma\bar{\phi}$ is $\ell_\sigma\ell$, and to make this no larger than the norm of $z\tau_\Delta$ on a block, we could choose $\ell_{\sigma,j} = \tilde{g}_j L^{[\phi]j}$. In addition, our choice of ℓ_σ must also be appropriate for the $\sigma\bar{\sigma}$ term which arises in V_{pt}. Our procedure is to take $q = 0$ in V. Thus, according to the flow of q given in [6, (3.35)], the $\sigma\bar{\sigma}$ term in V_{pt} is the increment $\lambda^2 C_{j+1;a,b}\sigma\bar{\sigma}$ (which is only nonzero above the coalescence scale j_{ab}). According to (1.33), with the above choice of ℓ_σ the norm of this term is of order $L^{-2[\phi]j}\ell_\sigma^2 = \tilde{g}_j^2$, and this is significantly smaller than the norm of the $\lambda\sigma\bar{\phi}$ term (which is good). However, a disadvantage of the choice $\ell_{\sigma,j} = \tilde{g}_j L^{[\phi]j}$ is that it would make the monomial $\sigma\bar{\sigma}\phi\bar{\phi}$ be marginal (scale invariant), hence in the range of Loc and thus in V_{pt}. This monomial only appears after the coalescence scale, and we would prefer it to be irrelevant. To achieve this, we decrease the size of ℓ_σ to the choice $\ell_\sigma = \tilde{g}_j 2^{(j-j_{ab})_+} L^{[\phi](j\wedge j_{ab})}$ made in (1.77). Then ℓ_σ grows as a power of L below the coalescence scale, but only by a power of 2 above the coalescence scale. This power of 2 plays a role in the proof of [4, Theorem 1.1].

3.3 The small parameter $\bar{\epsilon}$

For $j < N$, we define

$$\hat{\ell}_j^2 = \hat{\ell}_0^2 \ell_j^2 \|C_{(j+1)*}\|_{\Phi_j^+(\ell_j)}. \tag{3.17}$$

We choose $\hat{\ell}_0^2 = 100/c_G$, where $c_G = c(\alpha_G)$ is the constant of [13, Proposition 3.20] (this choice is useful in the proof of Lemma 6.1 below), so that

$$\|C_{(j+1)*}\|_{\Phi_j^+(\hat{\ell}_j)} = \|C_{(j+1)*}\|_{\Phi_j^+(\ell_j)} \frac{\ell_j^2}{\hat{\ell}_j^2} = \hat{\ell}_0^{-2} = \frac{1}{100} c_G. \tag{3.18}$$

Below the Ω-scale defined by (1.68), $\hat{\ell}_j$ and ℓ_j are of the same order of magnitude, but well above the Ω-scale $\hat{\ell}_j \ll \ell_j$. We use $\hat{\ell}_j$ in estimates involving integration, as a parameter which captures the size of the covariance effectively.

Let

$$\delta V = \theta V - V_{\mathrm{pt}} = \theta V - V_{\mathrm{pt},j+1}(V). \tag{3.19}$$

Recall the definition of $\bar{\epsilon}$ from (1.91). The following lemma justifies the notation used for $\bar{\epsilon}$, by showing that it provides an upper bound for δV. Its restriction to $j < N$ is to keep δV defined in (3.19).

Lemma 3.4. *Let $j < N$. There is an L-dependent constant $C_{\delta V}$ such that for all $V \in \bar{\mathcal{D}}_j$, and for $j_* = j$ or $j_* = j+1$,*

$$\max_{b \in \mathcal{B}_j} \|\delta V(b)\|_{T_{0,j_*}(\mathfrak{h}_{j_*} \sqcup \hat{\ell}_{j_*})} \leq C_{\delta V} \bar{\epsilon}. \tag{3.20}$$

Proof. We fix $j < N$, concentrate first on the case $j_* = j$, and drop subscripts j. We show that for $V \in \mathcal{Q}$ and $b \in \mathcal{B}_j$,

$$\|\delta V(b)\|_{T_0(\mathfrak{h} \sqcup \hat{\ell})} \prec_L \frac{\hat{\ell}}{\mathfrak{h}} \epsilon_V + \bar{\epsilon}^2. \tag{3.21}$$

This suffices, since the first term on the right-hand side of (3.21) obeys

$$\frac{\hat{\ell}}{\mathfrak{h}} \epsilon_V = \|C\|^{1/2}_{\Phi^+(\ell)} \frac{\ell}{\mathfrak{h}} \epsilon_V \prec \chi_j^{1/2} \frac{\ell}{\mathfrak{h}} \epsilon_V \prec_L \begin{cases} \chi_j^{1/2} \tilde{g} = \bar{\epsilon}(\ell) & \mathfrak{h} = \ell \\ \chi_j^{1/2} \tilde{g}^{1/4} = \bar{\epsilon}(h) & \mathfrak{h} = h. \end{cases} \tag{3.22}$$

This gives (3.20) and reduces the proof to showing (3.21).

We now prove (3.21). By (1.26) and (1.29), with $C = C_{j+1}$,

$$V_{\mathrm{pt}} - V = 2gC_{0,0}\tau - P. \tag{3.23}$$

Therefore, by definition of δV in (3.19) and by the triangle inequality,

$$\begin{aligned} \|\delta V(b)\|_{T_0(\mathfrak{h} \sqcup \hat{\ell})} &\leq \|\theta V(b) - V(b)\|_{T_0(\mathfrak{h} \sqcup \hat{\ell})} + \|V(b) - V_{\mathrm{pt}}(b)\|_{T_0(\mathfrak{h})} \\ &\leq \|\theta V(b) - V(b)\|_{T_0(\mathfrak{h} \sqcup \hat{\ell})} + \|C\|_{\Phi(\mathfrak{h})} \mathfrak{h}^2 \|2g\tau(b)\|_{T_0(\mathfrak{h})} + \|P(b)\|_{T_0(\mathfrak{h})}. \end{aligned} \tag{3.24}$$

For the first term on the right-hand side of (3.24), we use the triangle inequality to work term by term in the monomials in V. For example, the τ term makes a contribution

$$\|\nu(\theta\tau(b) - \tau(b))\|_{T_0(\mathfrak{h} \sqcup \hat{\ell})}. \tag{3.25}$$

After expansion in the fluctuation fields $\xi, \bar{\xi}, \eta, \bar{\eta}$, the difference $\theta\tau(b) - \tau(b)$ is given by a sum of products of fluctuation fields and $\phi, \bar{\phi}, \psi, \bar{\psi}$ fields, with each term containing two fields of which at least one is a fluctuation field. Thus it is bounded by $O(\hat{\ell}\mathfrak{h})$. The end result is a bound on $\|\theta V(b) - V(b)\|_{T_0(\mathfrak{h} \sqcup \hat{\ell})}$ equal to $\hat{\ell}/\mathfrak{h}$ times the $T_0(\mathfrak{h})$ semi-norm of the worst monomial in V (but without the $\sigma\bar{\sigma}$ term which cancels). This gives

$$\|\theta V(b) - V(b)\|_{T_0(\mathfrak{h} \sqcup \hat{\ell})} \prec \frac{\hat{\ell}}{\mathfrak{h}} \epsilon_V. \tag{3.26}$$

For the second term on the right-hand side of (3.24),

$$\|C\|_{\Phi} \mathfrak{h}^2 \|2g\tau(b)\|_{T_0} \prec \|C\|_{\Phi(\mathfrak{h})} \epsilon_V = \frac{\hat{\ell}^2}{\mathfrak{h}^2} \|C\|_{\Phi(\hat{\ell})} \epsilon_V. \tag{3.27}$$

For the last term, we use (1.93) to obtain

$$\|P(b)\|_{T_0} \prec_L \bar{\epsilon}^2. \tag{3.28}$$

The combination of the last three inequalities gives (3.21) and the proof for the case $j_* = j$ is complete.

Finally, for the case $j_* = j+1$, we start with the first line of (3.24) with norms at scale $j+1$. The norm of $V - V_{\mathrm{pt}}$ is bounded by its scale-j counterpart, by Lemma 3.2. In addition, (3.26) applies also at scale $j+1$, and this give the desired conclusion and completes the proof. ∎

4 Estimates on field polynomials

In this section, we prove the following proposition, which gives our main estimates on the field polynomials F, W, P. As usual, $\bar{\epsilon}$ depends on whether $\mathfrak{h} = \ell$ or $\mathfrak{h} = h$, as indicated in (1.91). Recall that P_j is defined for $0 \le j < N$, so there is no bound missing in (4.3).

Proposition 4.1. *For L sufficiently large and $V \in \bar{\mathcal{D}}_j$,*

$$\max_{B\in\mathcal{B}_j} \sum_{x\in B} \sum_{B'\in\mathcal{B}_j(\Lambda)} \|F_{\pi,C_{j*}}(V_x, V(B'))\|_{T_{0,j}(\mathfrak{h}_j)} \prec_L \bar{\epsilon}_j^2 \quad (j \le N), \tag{4.1}$$

$$\max_{B\in\mathcal{B}_j} \sum_{x\in B} \|W_j(V,x)\|_{T_{0,j}(\mathfrak{h}_j)} \prec_L \bar{\epsilon}_j^2 \quad (j \le N), \tag{4.2}$$

$$\max_{B\in\mathcal{B}_j} \sum_{x\in B} \|P_j(V,x)\|_{T_{0,j}(\mathfrak{h}_j)} \prec_L \bar{\epsilon}_j^2 \quad (j < N). \tag{4.3}$$

Remark 4.2. *Scale mismatch.* The bounds of Proposition 4.1 continue to hold if $T_{0,j\pm1}(\mathfrak{h}_{j\pm1})$ would be used on the left-hand sides instead of $T_{0,j}(\mathfrak{h}_j)$ (for indices that do not exceed the final scale). In fact, F and W are (non-local) polynomials of degree at most six, and P is a (local) polynomial of degree at most four. A change of ±1 in j in the evaluation of on of these T_0 semi-norms can therefore only give rise to a bounded power of L, and constants in (4.1)–(4.3) are permitted to depend on L.

We prepare for the proof in Section 4.1 with useful identities for P and W, and the proof is concluded in Section 4.2. The proof is based on a crucial contraction estimate from [14] for the operator Loc, which we recall below as Proposition 4.8.

4.1 Preliminary identities

The first lemma provides a formula for the expectation of F.

Lemma 4.3. *For polynomials A, B in the fields, and for covariances C, w,*

$$e^{\mathcal{L}_C} F_{\pi,w}(A,B) = F_{\pi,w+C}(e^{\mathcal{L}_C}A, e^{\mathcal{L}_C}B) - F_{\pi,C}(e^{\mathcal{L}_C}A, e^{\mathcal{L}_C}B). \tag{4.4}$$

Proof. By the definition of F in (1.17),

$$\begin{aligned} F_{w+C}(e^{\mathcal{L}_C}A, e^{\mathcal{L}_C}B) &= e^{\mathcal{L}_C}e^{\mathcal{L}_w}\big(e^{-\mathcal{L}_w}A\big)\big(e^{-\mathcal{L}_w}B\big) - (e^{\mathcal{L}_C}A)(e^{\mathcal{L}_C}B) \\ &= e^{\mathcal{L}_C}F_w(A,B) + e^{\mathcal{L}_C}(AB) - (e^{\mathcal{L}_C}A)(e^{\mathcal{L}_C}B) \\ &= e^{\mathcal{L}_C}F_w(A,B) + F_C(e^{\mathcal{L}_C}A, e^{\mathcal{L}_C}B). \end{aligned} \tag{4.5}$$

Rearrangement gives

$$e^{\mathcal{L}_C}F_w(A,B) = F_{w+C}(e^{\mathcal{L}_C}A, e^{\mathcal{L}_C}B) - F_C(e^{\mathcal{L}_C}A, e^{\mathcal{L}_C}B), \tag{4.6}$$

and, by (1.18) and the fact that the projection operators commute with $e^{\mathcal{L}_C}$, (4.6) extends to the same equation with F replaced by F_π. ∎

For the next lemma, we define

$$P_j(V'_x, V''_y) = \frac{1}{2}\mathrm{Loc}_x F_{\pi,w_{j+1}}(e^{\mathcal{L}_{j+1}}V'_x, e^{\mathcal{L}_{j+1}}V''_y) - \frac{1}{2}e^{\mathcal{L}_{j+1}}\mathrm{Loc}_x F_{\pi,w_j}(V'_x, V''_y) \quad (0 \le j < N-1), \tag{4.7}$$

$$W_j(V'_x, V''_y) = \frac{1}{2}(1-\mathrm{Loc}_x)F_{\pi,w_j}(V'_x, V''_y) \quad (1 \le j < N). \tag{4.8}$$

Both definitions will be extended to the final scale in Section 4.2.4, but this extension is not yet needed here. By definition, for $j < N$,

$$W_j(V,x) = \sum_{y\in\Lambda} W_j(V_x, V_y). \tag{4.9}$$

With the definition of $P_j(V)$ in (1.24), the next lemma shows that, for $j < N-1$,

$$P_j(V,x) = \sum_{y\in\Lambda} P_j(V_x, V_y). \tag{4.10}$$

For its proof, we observe that since $e^{\mathcal{L}_C}$ reduces the dimension of a monomial in the fields, $e^{\mathcal{L}_C} : \mathcal{V} \to \mathcal{V}$, and since Loc_X acts as the identity on $\mathcal{V}$, it follows that

$$\mathrm{Loc}_X e^{\mathcal{L}_C}\mathrm{Loc}_X = e^{\mathcal{L}_C}\mathrm{Loc}_X. \tag{4.11}$$

Lemma 4.4. *For $x, y \in \Lambda$, $0 \le j < N-1$, and for $V', V'' \in \mathcal{V}$,*

$$P_j(V'_x, V''_y) = \mathrm{Loc}_x\left(e^{\mathcal{L}_{j+1}}W_j(V'_x, V''_y) + \frac{1}{2}F_{\pi,C_{j+1}}(e^{\mathcal{L}_{j+1}}V'_x, e^{\mathcal{L}_{j+1}}V''_y)\right). \tag{4.12}$$

Proof. Consider first the case $j < N-1$. By definition of W_j in (1.20), the right-hand side of (4.12) can be rewritten as

$$\frac{1}{2}\mathrm{Loc}_x\big(e^{\mathcal{L}_{j+1}}(1-\mathrm{Loc}_x)F_{\pi,w_j}(V'_x, V''_y) + F_{\pi,C_{j+1}}(e^{\mathcal{L}_{j+1}}V'_x, e^{\mathcal{L}_{j+1}}V''_y)\big). \tag{4.13}$$

Application of (4.11) shows that (4.13) is equal to

$$\frac{1}{2}\mathrm{Loc}_x\big(e^{\mathcal{L}_{j+1}}F_{\pi,w_j}(V'_x,V''_y)+F_{\pi,C_{j+1}}(e^{\mathcal{L}_{j+1}}V'_x,e^{\mathcal{L}_{j+1}}V''_y)\big)-\frac{1}{2}e^{\mathcal{L}_{j+1}}\mathrm{Loc}_xF_{\pi,w_j}(V'_x,V''_y). \tag{4.14}$$

By (4.4), (4.14) is equal to

$$\frac{1}{2}\mathrm{Loc}_xF_{\pi,w_{j+1}}(e^{\mathcal{L}_{j+1}}V'_x,e^{\mathcal{L}_{j+1}}V''_y)-\frac{1}{2}e^{\mathcal{L}_{j+1}}\mathrm{Loc}_xF_{\pi,w_j}(V'_x,V''_y), \tag{4.15}$$

which is (4.7). ∎

The following lemma computes the expectation of W.

Lemma 4.5. *For $x,y\in\Lambda$, $j<N$, and for $V',V''\in\mathcal{V}$,*

$$e^{\mathcal{L}_j}W_{j-1}(V'_x,V''_y)=W_j(e^{\mathcal{L}_j}V'_x,e^{\mathcal{L}_j}V''_y)+P_{j-1}(V'_x,V''_y)-\frac{1}{2}F_{\pi,C_j}(e^{\mathcal{L}_j}V_x,e^{\mathcal{L}_j}V''_y). \tag{4.16}$$

Proof. By (4.8) and the formula (4.7) for P,

$$\begin{aligned}e^{\mathcal{L}_j}W_{j-1}(V'_x,V''_y)&=\frac{1}{2}e^{\mathcal{L}_j}F_{\pi,w_{j-1}}(V'_x,V''_y)-\frac{1}{2}e^{\mathcal{L}_j}\mathrm{Loc}_xF_{\pi,w_{j-1}}(V'_x,V''_y)\\&=\frac{1}{2}e^{\mathcal{L}_j}F_{\pi,w_{j-1}}(V'_x,V''_y)+P_{j-1}(V'_x,V''_y)\\&\quad-\frac{1}{2}\mathrm{Loc}_xF_{\pi,w_j}(e^{\mathcal{L}_j}V'_x,e^{\mathcal{L}_j}V''_y).\end{aligned} \tag{4.17}$$

Substitution of (4.4) into (4.17) gives

$$\begin{aligned}e^{\mathcal{L}_j}W_{j-1}(V'_x,V''_y)&=F_{\pi,w_j}(e^{\mathcal{L}_j}V'_x,e^{\mathcal{L}_j}V''_y)-F_{\pi,C_j}(e^{\mathcal{L}_j}V'_x,e^{\mathcal{L}_j}V''_y)\\&\quad+P_{j-1}(V'_x,V''_y)-\frac{1}{2}\mathrm{Loc}_xF_{\pi,w_j}(e^{\mathcal{L}_j}V'_x,e^{\mathcal{L}_j}V''_y),\end{aligned} \tag{4.18}$$

which is the same as (4.16). ∎

The next lemma applies Lemma 4.5 to obtain a formula that enables us to bound W recursively, in Proposition 4.10 below.

Lemma 4.6. *For $x,y\in\Lambda$, $j<N$, and $V',V''\in\mathcal{V}$,*

$$W_j(V'_x,V''_y)=(1-\mathrm{Loc}_x)\Big(e^{\mathcal{L}_j}W_{j-1}(e^{-\mathcal{L}_j}V'_x,e^{-\mathcal{L}_j}V''_y)+\frac{1}{2}F_{\pi,C_j}(V'_x,V''_y)\Big). \tag{4.19}$$

Proof. The equalities

$$\begin{aligned}
W_j(e^{\mathcal{L}_j}V'_x, e^{\mathcal{L}_j}V''_y) &= e^{\mathcal{L}_j}W_{j-1}(V'_x, V''_y) - P_{j-1}(V'_x, V''_y) + \frac{1}{2}F_{\pi,C_j}(e^{\mathcal{L}_j}V_x, e^{\mathcal{L}_j}V''_y) \\
&= e^{\mathcal{L}_j}W_{j-1}(V'_x, V''_y) - \frac{1}{2}\mathrm{Loc}_x F_{\pi,w_j}(e^{\mathcal{L}_j}V'_x, e^{\mathcal{L}_j}V''_y) + \frac{1}{2}e^{\mathcal{L}_j}\mathrm{Loc}_x F_{\pi,w_{j-1}}(V'_x, V''_y) \\
&+ \frac{1}{2}F_{\pi,C_j}(e^{\mathcal{L}_j}V_x, e^{\mathcal{L}_j}V''_y) \\
&= e^{\mathcal{L}_j}W_{j-1}(V'_x, V''_y) + \frac{1}{2}e^{\mathcal{L}_j}\mathrm{Loc}_x F_{\pi,w_{j-1}}(V'_x, V''_y) + \frac{1}{2}F_{\pi,C_j}(e^{\mathcal{L}_j}V'_x, e^{\mathcal{L}_j}V''_y) \\
&- \frac{1}{2}\mathrm{Loc}_x F_{\pi,C_j}(e^{\mathcal{L}_j}V'_x, V''_y) - \frac{1}{2}\mathrm{Loc}_x e^{\mathcal{L}_j}F_{\pi,w_{j-1}}(V'_x, V''_y) \\
&= e^{\mathcal{L}_j}W_{j-1}(V'_x, V''_y) + \frac{1}{2}(1-\mathrm{Loc}_x)F_{\pi,C_j}(e^{\mathcal{L}_j}V'_x, e^{\mathcal{L}_j}V''_y) \\
&+ \frac{1}{2}\mathrm{Loc}_x e^{\mathcal{L}_j}\mathrm{Loc}_x F_{\pi,w_{j-1}}(V'_x, V''_y) - \frac{1}{2}\mathrm{Loc}_x e^{\mathcal{L}_j}F_{\pi,w_{j-1}}(V'_x, V''_y) \qquad (4.20)
\end{aligned}$$

give the desired result. The first equality is (4.16), the second follows from the formula for P in (4.7), the third uses (4.4), and for the last we used (4.11) to insert an operator Loc_x acting on the second term of the third right-hand side. ∎

4.2 Proof of Proposition 4.1

We now prove the estimates on F, W, P stated in Proposition 4.1. We first consider F, then recall the crucial contraction estimate from [14] concerning the operator Loc, then apply the contraction estimate to obtain bounds on W and P.

4.2.1 Bound on F

We now prove the bound (4.1) on F.

Operator bounds on the Laplacian as a map on T_ϕ are given in [13, Proposition 3.18], which asserts that the operators $\mathcal{L}_C$ and $e^{\pm\mathcal{L}_C}$, restricted to the subspace of $\mathcal{N}$ consisting of polynomials of degree A with semi-norm $\|\cdot\|_{T_\phi}$, are bounded operators whose norms obey

$$\|\mathcal{L}_C\| \le A^2\|C\|_\Phi, \qquad \|e^{\pm\mathcal{L}_C}\| \le e^{A^2\|C\|_\Phi}. \qquad (4.21)$$

The above operator norms are for operators acting on T_ϕ, with the scale fixed.

Let $Y(C,x) = \{y : C_{x,y} \neq 0\}$. Recall (1.2), which implies that the diameter and volume of $Y(C_k, x)$ obey

$$\mathrm{diam}\,(Y(C_k,x)) \le L^k, \qquad |Y(C_k,x)| \le L^{dk}. \qquad (4.22)$$

We recall the definition of $\overset{\leftrightarrow}{\mathcal{L}}_w$ from [6, (5.23)], and also recall [6, Lemma 5.6], which asserts that for V', V'' of degree at most A,

$$F_w(V'_x, V''_y) = \sum_{n=1}^{A}\frac{1}{n!}(\overset{\leftrightarrow}{\mathcal{L}}_w)^n(V'_x V''_y). \qquad (4.23)$$

Lemma 4.7. *Suppose that* $\|C\|_\Phi \le 1$. *Then for* $x, y \in \Lambda$ *and* $V', V'' \in \mathcal{V}$,

$$\|F_{\pi,C}(V'_x, V''_y)\|_{T_0} \prec \|C\|_\Phi \|V'_x\|_{T_0} \|V''_y\|_{T_0} \mathbb{1}_{y \in Y(C,x)}. \tag{4.24}$$

Also, $F_{\pi,C}(V'_x, V''_y) \in \mathcal{N}(Y(C,x))$ *and* $\sum_{y\in\Lambda} W_w(V'_x, V''_y) \in \mathcal{N}(Y(w,x))$.

Proof. By (1.18), it follows from (4.23) that $F_{\pi,C}(V'_x, V''_y) \in \mathcal{N}(Y(C,x))$. It then follows from (1.20) that $W_w(V', V'', \{x\}) \in \mathcal{N}(Y(w,x))$.

Now we prove (4.24). We have already shown that the left-hand side is zero for $y \notin Y(C,x)$, so it suffices to prove (4.24) without the factor $\mathbb{1}_{y\in Y(C,x)}$. Furthermore, by (1.18), it is enough to prove (4.24) with $F_{\pi,C}$ replaced by F_C. For $t \ge 0$, let

$$F(t) = e^{\mathcal{L}_{tC}} \left((e^{-\mathcal{L}_{tC}} V'_x)(e^{-\mathcal{L}_{tC}} V''_y) \right). \tag{4.25}$$

Since V', V'' are polynomials in fields, by expanding each of the exponentials we find that $F(t)$ is a polynomial $\sum_{n=0}^m F_n t^n$, for some finite m. According to the second inequality of (4.21), there is a $k > 0$ determined by $\|tC\|_\Phi$ such that

$$\sum_{n=0}^m \|F_n\|_{T_0} |t|^n \le k \|V'_x\|_{T_0} \|V''_y\|_{T_0}. \tag{4.26}$$

Although k depends on $\|tC\|_\Phi$, it is uniform for $\|tC\|_\Phi \le 1$. By (1.17),

$$F_C(V'_x, V''_y) = F(1) - F(0) = \sum_{n=1}^m F_n. \tag{4.27}$$

Therefore, taking $t = \|C\|_\Phi^{-1} \ge 1$, we obtain

$$\|F_C(V'_x, V''_y)\|_{T_0} \le \sum_{n=1}^m \|F_n\|_{T_0} \le \frac{1}{t} \sum_{n=1}^m \|F_n\|_{T_0} t^n \le k \|C\|_\Phi \|V'_x\|_{T_0} \|V''_y\|_{T_0}, \tag{4.28}$$

which completes the proof. ∎

To estimate the covariance of C_j, we use the fact that for $j \le N$,

$$\|C_{j*}\|_{\Phi_j(\mathfrak{h}_k)} \prec_L \begin{cases} \chi_j & \mathfrak{h}_j = \ell_j \\ \chi_j \tilde{g}_j^{1/2} & \mathfrak{h}_j = h_j. \end{cases} \tag{4.29}$$

The case $\mathfrak{h}_j = \ell_j$ follows immediately from (1.72), and the case $\mathfrak{h}_j = h_j$ follows from

$$\|C_{j*}\|_{\Phi_j(h_j)} = \left(\frac{\ell_j}{h_j}\right)^2 \|C_{j*}\|_{\Phi_j(\ell_j)} = \left(\frac{\ell_0}{k_0}\right)^2 \tilde{g}_j^{1/2} \|C_{j*}\|_{\Phi_j(\ell_j)} \prec_L \tilde{g}_j^{1/2} \chi_j. \tag{4.30}$$

Proof of (4.1). Let $1 \le j \le N$. Summation of (4.24) gives, for any $B \in \mathcal{B}_j$, the upper bound

$$\sum_{x\in B}\sum_{y\in\Lambda} \|F_{\pi,C_{j*}}(V'_x, V'_y)\|_{T_{0,j}(\mathfrak{h}_j)} \prec \|C_j\|_{\Phi_j(\mathfrak{h}_j)}\epsilon_{V',j}\epsilon_{V'',j}. \tag{4.31}$$

We set $V' = V'' = V$ in (4.31). Since $V \in \bar{\mathcal{D}}_j$, $\epsilon_{V,j}$ is bounded by a multiple of $\tilde{g}_j$ for $\mathfrak{h} = \ell$, and of 1 for $\mathfrak{h} = h$. With (4.29), this gives

$$\max_{B\in\mathcal{B}_j}\sum_{x\in B}\sum_{y\in\Lambda} \|F_{\pi,C_{j*}}(V_x, V_y)\|_{T_{0,j}} \prec_L \bar{\epsilon}^2, \tag{4.32}$$

which is the desired estimate (4.1). ∎

4.2.2 Loc and the crucial contraction

It is shown in [14, Proposition 1.18] (with $R = L^{-j}$) that Loc_X is a bounded operator on T_0 in the sense that if $F \in \mathcal{N}(X)$ then

$$\|\mathrm{Loc}_X F\|_{T_0} \le \bar{C}'\|F\|_{T_0}, \tag{4.33}$$

where $\bar{C}'$ depends on $L^{-j}\mathrm{diam}(X)$. We also recall [14, Proposition 1.19], which is the crucial contraction estimate which we state here as follows. As in [14, Definition 1.17], we use the notation $X(\varnothing) = X$, $X(a) = X\cap\{a\}$, $X(b) = X\cap\{b\}$, and $X(ab) = X\cap\{a,b\}$. As discussed in Section 1.1.3, $d_+ = d$ on $\mathcal{N}^\varnothing$ and $d_+ = [\phi]$ on each of $\mathcal{N}^a, \mathcal{N}^b, \mathcal{N}^{ab}$. For $\alpha, \beta \in \{\varnothing, a, b, ab\}$, we define $d'_\alpha = d_\alpha + 1$, and

$$\gamma_{\alpha,\beta} = (L^{-d'_\alpha} + L^{-(A+1)[\phi]})\left(\frac{\mathfrak{h}'_\sigma}{\mathfrak{h}_\sigma}\right)^{|\alpha\cup\beta|}. \tag{4.34}$$

Proposition 4.8. *Let $A < p_{\mathcal{N}}$ be a positive integer, and let $\varnothing \ne Y \subset X \in \mathcal{P}_j$. Let $F_1 \in \mathcal{N}(X)$, and let $F_2 \in \mathcal{N}(Y)$ with $\pi_\alpha F_2 = 0$ when $Y(\alpha) = \varnothing$. Let $F = F_1(1 - \mathrm{Loc}_Y)F_2$. Let T'_ϕ denote the $T_{\phi,j+1}(c\mathfrak{h}_{j+1})$ semi-norm for any fixed $c \ge 1$, and let T_ϕ denote the $T_{\phi,j}(\mathfrak{h}_j)$ semi-norm. Then*

$$\begin{aligned}\|F\|_{T'_\phi} \le \bar{C} &\sum_{\alpha,\beta=\varnothing,a,b,ab} \gamma_{\alpha,\beta}\,(1 + \|\phi\|_{\Phi'})^{A+d_\alpha/[\phi]+1} \\ &\times \sup_{0\le t\le 1}\big(\|\pi_\beta F_1\pi_\alpha F_2\|_{T_{t\phi}} + \|\pi_\beta F_1\|_{T_{t\phi}}\|\pi_\alpha F_2\|_{T_0}\big)\|\sigma^{\alpha\cup\beta}\|_{T_0},\end{aligned} \tag{4.35}$$

where $\bar{C}$ depends on c and $L^{-j}\mathrm{diam}(X)$.

As a corollary, we specialise to our particular setting to obtain the following proposition. We state Proposition 4.9 in a more general form than is needed to bound W, but the additional generality is used in the proof of Proposition 2.8.

Proposition 4.9. *Let $d \ge 4$, $A = \lceil 2(d+1)/(d-2)\rceil$, and assume that $p_{\mathcal{N}} > A$. Let $\varnothing \ne Y \subset X \in \mathcal{P}_j$. Let $F_1 \in \mathcal{N}(X)$, let $F_2 \in \mathcal{N}(Y)$ with $\pi_\alpha F_2 = 0$ when $Y(\alpha) = \varnothing$, and let $F = F_1(1 - \mathrm{Loc}_Y)F_2$. Suppose that each of F_1, F_2, F_1F_2 has no component in $\mathcal{N}_{ab}$ unless $j \ge j_{ab}$ (recall* (1.13)*). Let T'_ϕ denote the $T_{\phi,j+1}(c\mathfrak{h}_{j+1})$ semi-norm for some fixed $c \ge 1$, and let T_ϕ denote the $T_{\phi,j}(\mathfrak{h}_j)$ semi-norm. There is a constant $\bar{C}$ depending on c and $L^{-j}\mathrm{diam}(X)$ such that*

$$\|F\|_{T'_\phi} \le \bar{C}\gamma\,(1 + \|\phi\|_{\Phi'})^{A+d+1} \sup_{0\le t\le 1}\big(\|F_1F_2\|_{T_{t\phi}} + \|F_1\|_{T_{t\phi}}\|F_2\|_{T_0}\big), \tag{4.36}$$

with

$$\gamma = \gamma(Y) = L^{-d-1} + L^{-1}\mathbb{1}_{Y\cap\{a,b\}\neq\varnothing}. \tag{4.37}$$

Moreover, if $\pi_* F_2 = 0$ *then we can replace* (4.37) *by* $\gamma = L^{-d-1}$

Proof. In our setting, $d_\varnothing = d$, and $d_\alpha = 1$ for $\alpha = a, b, ab$. Also, $[\phi] = \frac{d-2}{2} \geq 1$ for all α. In particular, $A + d_\alpha/[\phi] + 1 \leq A + d + 1$. Our choice of A ensures that $(A+1)[\phi] \geq d + 1 \geq d_\alpha + 1$ for all α. By (1.78),

$$\frac{\mathfrak{h}_{\sigma,j+1}}{\mathfrak{h}_{\sigma,j}} \leq \text{const} \begin{cases} L^{+[\phi]} & j < j_{ab} \\ 1 & j \geq j_{ab}. \end{cases} \tag{4.38}$$

By assumption, when $|\alpha \cup \beta| = 2$ we can use the $j \geq j_{ab}$ version of the above bound. Also by assumption, for $\alpha = a, b, ab$ we have $\pi_\alpha F_2 = 0$ when $Y \cap \{a, b\} = \varnothing$. Taking these points into account, from (4.34) we obtain

$$\gamma_{\alpha,\beta} \leq 2 \begin{cases} L^{-d-1} & |\alpha \cup \beta| = 0 \\ L^{-1}\mathbb{1}_{Y\cap\{a,b\}\neq\varnothing} & |\alpha \cup \beta| = 1, 2. \end{cases} \tag{4.39}$$

This shows that $\gamma_{\alpha,\beta} \leq 2\gamma$ uniformly in α, β. It follows from (3.1) that

$$\big(\|\pi_\beta F_1 \pi_\alpha F_2\|_{T_{t\phi}} + \|\pi_\beta F_1\|_{T_{t\phi}} \|\pi_\alpha F_2\|_{T_0}\big) \|\sigma^{\alpha\cup\beta}\|_{T_0} \leq \|F_1 F_2\|_{T_{t\phi}} + \|F_1\|_{T_{t\phi}} \|F_2\|_{T_0}. \tag{4.40}$$

Together with Proposition 4.8, these facts give the desired estimate and the proof is complete. ∎

4.2.3 Bound on W

We now prove (4.2) for $j < N$, beginning with the following proposition, whose proof requires our assumption that L is large. We defer the case $j = N$ of (4.2) (and also of (4.41)) to Sections 4.2.4–4.2.5.

Proposition 4.10. *Let* $j < N$. *In general,* $\pi_{ab} W_j = 0$. *Let* $V', V'' \in \mathcal{V}$. *Suppose there is a sequence* v'_k *with* $v'_{k-1} \prec v'_k$ *for all* $k \leq j$, *such that* $\max_{B\in\mathcal{B}_k} \sum_{x\in B} \|V'_x\|_{T_{0,k}} \leq v'_k$, *and similarly for* V''. *Then there is a constant* c *such that*

$$\max_{B\in\mathcal{B}_j} \sum_{x\in B} \sum_{y\in\Lambda} \|W_j(V'_x, V''_y)\|_{T_{0,j}(\mathfrak{h}_j)} \leq c\chi_j \left(\frac{\ell_j}{\mathfrak{h}_j}\right)^2 v'_j v''_j. \tag{4.41}$$

Proof. In W, we can exclude the $\sigma\bar\sigma$ terms in each of V', V'' since these contribute zero to F. Thus the only possible $\sigma\bar\sigma$ contribution to W can be due to the contribution to F due to a contraction of $\sigma\bar\phi_a$ with $\bar\sigma\phi_b$. Such a contraction contains no boson or fermion fields, so is annihilated by $1 - \mathrm{Loc}_{\{x\}}$. This proves that $\pi_{ab} W = 0$, and it remains to prove (4.41).

We prove (4.41), by induction on j. Our induction hypothesis is that (4.41) holds for $j-1$, and we use this to prove that it also holds for j. Initially $W_0 = 0$, so it is trivial to begin the induction. The starting point is Lemma 4.6, which implies that

$$W_j(V'_x, V''_y) = (1 - \mathrm{Loc}_x)\Big(e^{\mathcal{L}_j} W_{j-1}(e^{-\mathcal{L}_j} V'_x, e^{-\mathcal{L}_j} V''_y) + \frac{1}{2} F_{\pi,C_j}(V'_x, V''_y)\Big). \tag{4.42}$$

We estimate (4.42) using the triangle inequality on the right-hand side, retaining the cancellation in $1 - \mathrm{Loc}_{\{x\}}$ for the first term but not for the second. With (4.33), this gives

$$\begin{aligned}\|W_j(V'_x, V''_y)\|_{T_{0,j}} \le \|(1 - \mathrm{Loc}_x)e^{\mathcal{L}_j} W_{j-1}(e^{-\mathcal{L}_j} V'_x, e^{-\mathcal{L}_j} V''_y)\|_{T_{0,j}} \\ + \frac{1}{2}(1 + \bar{C})\|F_{\pi,C_j}(V'_x, V''_y)\|_{T_{0,j}}. \end{aligned} \tag{4.43}$$

The constant $\bar{C}$ is independent of j, as a consequence of (4.22) together with the fact that $F_{\pi,C_j}(V'_x, V''_y) \in \mathcal{N}(Y(C_j, x))$ by Lemma 4.7.

We begin with the second term on the right-hand side of (4.43). After application of Lemma 4.7 and (4.29), and summation over x, y, we find that there is a constant f such that

$$\frac{1}{2}(1 + \bar{C}) \max_{B \in \mathcal{B}_j} \sum_{x \in B} \sum_{y \in \Lambda} \|F_{\pi,C_j}(V'_x, V''_y)\|_{T_{0,j}} \le \bar{f} \chi_j \left(\frac{\ell_j}{\mathfrak{h}_j}\right)^2 v'_j v''_j. \tag{4.44}$$

For the first term on the right-hand side, we apply Proposition 4.9 with $F_1 = 1$ and $F_2 = e^{\mathcal{L}_j} W_{j-1}(e^{-\mathcal{L}_j} V'_x, e^{-\mathcal{L}_j} V''_y)$. Note that, as required by the hypotheses of Proposition 4.9, $\pi_* F_2 = 0$ unless $x \in \{a, b\}$; this is a consequence of the careful definition of $F_{\pi,C}$ in (1.18), which ensures that if one of $\pi_* V'$ or $\pi_* V''$ is nonzero then $\pi_* V'$ must be nonzero. The application of Proposition 4.9 gives the estimate

$$\|(1 - \mathrm{Loc}_x)F_2\|_{T_{0,j}} \le \bar{C}\gamma_x \|F_2\|_{T_{0,j-1}}, \tag{4.45}$$

with

$$\gamma_x = L^{-d-1} + L^{-1} \mathbb{1}_{x \in \{a,b\}}, \tag{4.46}$$

and with a scale-independent constant $\bar{C}$ since $F_2 \in \mathcal{N}(Y(C_{j-1}, x))$ by Lemma 4.7. The operators $e^{\pm \mathcal{L}_j}$ are bounded on $T_{0,j-1}$, by (4.21) and the fact that

$$\|C_j\|_{\Phi_{j-1}(\mathfrak{h}_{j-1})} \le \|C_j\|_{\Phi_j(\mathfrak{h}_{j-1})} = (\ell_j/\mathfrak{h}_{j-1})^2 \|C_j\|_{\Phi_j(\ell_j)} \le (\ell_j/\mathfrak{h}_{j-1})^2 \mathfrak{c} \le 1 \tag{4.47}$$

using (1.78) and (1.72). Thus, by the induction hypothesis, there is a constant $\bar{A}$ such that

$$\max_{B \in \mathcal{B}_j} \sum_{x \in B} \sum_{y \in \Lambda} \gamma_x \|F_2\|_{T_{0,j}} \le \bar{A} c L^{-1} \chi_j \left(\frac{\ell_j}{\mathfrak{h}_j}\right)^2 v'_j v''_j, \tag{4.48}$$

where we have used the fact that B contains L^d blocks of scale $j-1$, our assumption on the sequences v'_k and v''_k, and that the factors involving χ and $\ell/\mathfrak{h}$ change only by a constant factor under a single advance of scale.

The combination of (4.44), (4.45) and (4.48) gives

$$\max_{B \in \mathcal{B}_j} \sum_{x \in B} \sum_{y \in \Lambda} \|W_j(V'_x, V''_y)\|_{T_{0,j}(\mathfrak{h}_j)} \le (\bar{C}\bar{A}cL^{-1} + f)\chi_j \left(\frac{\ell_j}{\mathfrak{h}_j}\right)^2 v'_j v''_j \mathbb{1}_{y \in Y(C_k, x)}. \tag{4.49}$$

We require that $L > \bar{C}\bar{A}$ (which we can do in view of our general hypothesis that L is large enough). Then (4.49) advances the induction with the choice $c = f/(1 - \bar{C}\bar{A}L^{-1})$, since this choice gives $\bar{C}\bar{A}cL^{-1} + f = c$. This completes the proof. ∎

Proof of (4.2) *for* $j < N$. Let $j < N$. For $V \in \bar{\mathcal{D}}_j$, by direct computation as in the proof of Proposition 1.4, we find that, for any $k \le j$ and $b \in \mathcal{B}_k$, $\sum_{x \in B} \|V_x\|_{T_{0,k}}$ is bounded above by a multiple of $\tilde{g}_j$ for $\mathfrak{h} = \ell$, and of $\tilde{g}_j / \tilde{g}_k$ for $\mathfrak{h} = h$. We apply Proposition 4.10 with these two choices for v_k, which do obey its hypothesis by (1.76). This gives

$$\sum_{x \in B} \|W_j(V, x)\|_{T_{0,j}} \prec_L \begin{cases} \chi_j \tilde{g}_j^2 & \mathfrak{h} = \ell \\ \chi_j \tilde{g}_j^{1/2} & \mathfrak{h} = h. \end{cases} \tag{4.50}$$

The right-hand side is $\bar{\epsilon}_j^2$ and this completes the proof. ∎

4.2.4 Bound on P

We now prove (4.3), and also prove the case $j = N$ of (4.2).

Proof of (4.3). We first consider $j < N - 1$, and recall from Lemma 4.4 that

$$P_j(V'_x, V''_y) = \mathrm{Loc}_x \left(e^{\mathcal{L}_{j+1}} W_j(V'_x, V''_y) + \frac{1}{2} F_{\pi, C_{j+1}}(e^{\mathcal{L}_{j+1}} V'_x, e^{\mathcal{L}_{j+1}} V''_y) \right). \tag{4.51}$$

We bound the operator norms of Loc_x and $e^{\mathcal{L}_{j+1}}$ as discussed previously (using (4.47)), and apply (4.31) and (4.41), to conclude that under the same hypothesis on V', V'' as in Proposition 4.10,

$$\max_{B \in \mathcal{B}_j} \sum_{x \in B} \sum_{y \in \Lambda} \|P_j(V'_x, V''_y)\|_{T_{0,j}(\mathfrak{h}_j)} \prec \chi_j \left(\frac{\ell_j}{\mathfrak{h}_j} \right)^2 v'_j v''_j. \tag{4.52}$$

Then we set $V' = V'' = V \in \bar{\mathcal{D}}_j$ and as in the proof of (4.50) we obtain

$$\max_{B \in \mathcal{B}_j} \sum_{x \in B} \|P_j(V, x)\|_{T_{0,j}(\mathfrak{h}_j)} \prec_L \bar{\epsilon}_j^2 \tag{4.53}$$

as desired. This completes the proof of (4.3) for $j < N - 1$.

As discussed in Section 1.1.5, our definition of P_{N-1} is designed so that P_{N-1} for the torus of scale N is the same local polynomial as $P_{N-1,N+1}$ on the torus of scale $N + 1$. Consequently we can apply (4.3) on the torus of scale $N + 1$ to obtain the desired estimate (4.3) on P_{N-1}. ∎

Proof of (4.2) *for* $j = N$. According to (1.31),

$$W_N(V, x) = e^{\mathcal{L}_{N,N}} W_{N-1}(e^{-\mathcal{L}_{N,N}} V, x) - P_{N-1}(e^{-\mathcal{L}_{N,N}} V, x) + \frac{1}{2} F_{\pi, C_{N,N}}(V_x, V(\Lambda)). \tag{4.54}$$

This obeys (4.2) by using (1.72) and (4.21) together with the estimates on W_{N-1}, P_{N-1}, $F_{\pi, C_{N,N}}$ obtained above. ∎

4.2.5 Auxiliary estimates on W

In (1.29), we defined $P_{N-1}(V)$ to be equal to the common value that (1.24) would give on any torus of scale larger than N. Similarly, we extend the definition of $P_j(V'_x, V''_y)$ to $j = N-1$ by defining it to be the common value of the right-hand side of (4.7), with $j = N-1$, on any torus of scale larger than N. In addition, we adapt the identity (4.16) to define $W_N(V'_x, V''_y)$ (which has not yet been defined for distinct V', V'') as

$$W_N(V'_x, V''_y) = e^{\mathcal{L}_{N,N}} W_{N-1}(e^{-\mathcal{L}_{N,N}} V'_x, e^{-\mathcal{L}_{N,N}} V''_y) - P_{N-1}(e^{-\mathcal{L}_{N,N}} V'_x, e^{-\mathcal{L}_{N,N}} V''_y) + \frac{1}{2} F_{\pi, C_{N,N}}(V_x, V''_y). \tag{4.55}$$

Then from (4.54) we see that the identity (4.9) extends to scale $j = N$:

$$W_N(V, x) = \sum_{y \in \Lambda} W_N(V_x, V_y). \tag{4.56}$$

Also, the estimate (4.41) of Proposition 4.10 now extends to scale N. To see this, we use the definition (4.55), the fact that $e^{\pm \mathcal{L}_{N,N}}$ is a bounded operator, the bounds on W_{N-1} and F obtained previously, and finally the fact that (4.52) extends to its final scale $N-1$ by application of (4.52) on a larger torus.

The next lemma provides a concrete upper bound on $W_j(V', V'')$ when observables are absent.

Lemma 4.11. *Suppose that $V', V'' \in \pi_\varnothing \mathcal{V}$, and let $|V|_j = \max\{|g|, L^{2j}|\nu|, |z|, |y|\}$. For $j \le N$,*

$$\max_{B \in \mathcal{B}_j} \sum_{x \in B} \sum_{y \in \Lambda} \|W_j(V'_x, V''_y)\|_{T_{0,j}(\ell_j)} \prec_L \chi_j |V'|_j |V''|_j. \tag{4.57}$$

Proof. Let $v_k = L^{dk} \|V_x\|_{T_{0,k}(\ell_k)}$. Direct computation of the $T_{0,k}(\ell_k)$ norm shows that $v_k \prec |V|_k \le |V|_j$ for $k \le j$. Then Proposition 4.10 (extended as noted above to include $j = N$) gives a bound on the left-hand side of (4.57) of order v_j^2, as desired. ∎

Finally, the next lemma provides estimates for later use.

Lemma 4.12. *For $j+1 \le N$, $B \in \mathcal{B}_j$, and $V \in \bar{\mathcal{D}}_j$,*

$$\|W_{j+1}(e^{\mathcal{L}_{j+1}} V, B) - W_{j+1}(V_{\mathrm{pt}}, B)\|_{T_{0,j+1}} \prec_L \bar{\epsilon}_j^3. \tag{4.58}$$

For $j \le N$, $B \in \mathcal{B}_j$, $V \in \bar{\mathcal{D}}_j$, and for $Q \in \mathcal{Q}$ with $\|Q(B)\|_{T_{0,j}} \prec \bar{\epsilon}_j$,

$$\|W_j(Q(B), V(\Lambda))\|_{T_0,j} \prec_L \begin{cases} \bar{\epsilon}_j^2 & \mathfrak{h} = \ell \\ \bar{\epsilon}_j^2 \tilde{g}_j^{1/4} & \mathfrak{h} = h, \end{cases} \tag{4.59}$$

$$\|W_j(Q(B), Q(\Lambda))\|_{T_0,j} \prec_L \begin{cases} \bar{\epsilon}_j^2 & \mathfrak{h} = \ell \\ \bar{\epsilon}_j^2 \tilde{g}_j^{1/2} & \mathfrak{h} = h. \end{cases} \tag{4.60}$$

Proof. By linearity and the triangle inequality,

$$\|W_{j+1}(e^{\mathcal{L}_{j+1}}V) - W_{j+1}(V_{\mathrm{pt}})\|_{T_{0,j+1}} \le \|W_{j+1}(P, e^{\mathcal{L}_{j+1}}V)\|_{T_{0,j+1}} + \|W_{j+1}(V_{\mathrm{pt}}, P)\|_{T_{0,j+1}}. \tag{4.61}$$

We apply Proposition 1.4, use Proposition 4.1 to see that for $B_k \in \mathcal{B}_k$ it is the case that $\sum_{x\in B_k} \|P_x\|_{T_{0,k}} \prec_L \bar{\epsilon}_k^2$, and then apply Proposition 4.10 (including its extension to scale N), to see that

$$\|W_{j+1}(e^{\mathcal{L}_{j+1}}V) - W_{j+1}(V_{\mathrm{pt}})\|_{T_{0,j+1}} \prec_L \chi_j(\ell_j/\mathfrak{h}_j)^2\bar{\epsilon}_j^2 \times \begin{cases} \tilde{g}_j & \mathfrak{h} = \ell \\ 1 & \mathfrak{h} = h \end{cases} \prec \bar{\epsilon}_j^3, \tag{4.62}$$

as required. For (4.59)–(4.60), a similar calculation, using $\epsilon_Q \prec \bar{\epsilon}$ (by Lemma 3.1 and assumption) gives the desired result. This completes the proof. ∎

5 Proof of Propositions 2.1–2.4

In this section, we prove Propositions 2.1–2.4. Attention is restricted here to $d = 4$.

We begin by proving estimates on $\mathcal{I} = e^{-V}$ of (1.21). Since norms in the global space $\Phi = \Phi(\Lambda)$ can be replaced in upper bounds by the local space $\Phi(X)$ whenever an element of $\mathcal{N}(X)$ is being estimated (as discussed around [13, (3.38)]), we sometimes write simply Φ rather than $\Phi(X)$. However, decay estimates (such as (5.3) below) must always be stated in localised form.

Temporarily, we write a_0, b_0 (rather than the usual a, b) for the points where observables are located in V, and instead we use b for a block in $\mathcal{B}_{j-1}$. Also, we write

$$\epsilon_V(b) = \begin{cases} L^{-d}\epsilon_{V_\varnothing} & \{a_0, b_0\} \cap b = \varnothing \\ L^{-d}\epsilon_{V_\varnothing} + 2|\lambda|\mathfrak{h}\mathfrak{h}_\sigma + |q|\mathfrak{h}_\sigma^2 & \{a_0, b_0\} \cap b \neq \varnothing, \end{cases} \tag{5.1}$$

as opposed to ϵ_V which always includes the contribution from the observables.

Proposition 5.1. *Let $j \le N$. Let $V \in \mathcal{Q}$ with $0 \le |\mathrm{Im}g| \le \frac{1}{2}\mathrm{Re}g$.*
(i) For $b \in \mathcal{B}_{j-1}$,

$$\|\mathcal{I}(b)\|_{T_\phi} \le e^{O(\epsilon_V(b))(1+\|\phi\|_\Phi^2)}. \tag{5.2}$$

(ii) Fix any $q \ge 0$. Suppose that $\epsilon_V \le C\epsilon_{g\tau^2}$ for some $C > 0$. For $B \in \mathcal{B}_j$, and $X \in \mathcal{S}_{j-1}(B)$ or $X = \varnothing$,

$$\|\mathcal{I}(B \setminus X)\|_{T_\phi} \le e^{O(1+q^2)\epsilon_V} e^{-q\epsilon_{g\tau^2}\|\phi\|^2_{\Phi(B^\square)}} e^{O(1+q)\epsilon_V\|\phi\|^2_{\tilde{\Phi}(B^\square)}}. \tag{5.3}$$

Proof. (i) We write $V = g\tau^2 + Q$. By [13, Proposition 3.9] (with $q_2 = 0$),

$$\|e^{-g\tau_x^2}\|_{T_\phi} \le e^{O(|g|\mathfrak{h}^4)} = e^{O(L^{-dj}\epsilon_{g\tau^2})}. \tag{5.4}$$

By the product property and (1.80),

$$\|e^{-g\tau^2(b)}\|_{T_\phi} \le \prod_{x\in b} \|e^{-g\tau_x^2}\|_{T_\phi} \le e^{O(L^{-d}\epsilon_{g\tau^2})}. \tag{5.5}$$

Also, since Q is quadratic, from [13, Proposition 3.10] and (1.79) we obtain

$$\|Q(b)\|_{T_\phi} \leq \|Q(b)\|_{T_0}(1 + \|\phi\|_\Phi^2) \leq \epsilon_V(b)(1 + \|\phi\|_\Phi^2). \tag{5.6}$$

Therefore, by the power series expansion of the exponential and the product property,

$$\|e^{-Q(b)}\|_{T_\phi} \leq e^{\|Q(b)\|_{T_\phi}} \leq e^{\epsilon_V(b)(1+\|\phi\|_\Phi^2)}. \tag{5.7}$$

With the product property, (5.2) then follows from (5.5), (5.7), and the fact that $\epsilon_{g\tau^2} \leq \epsilon_{V_\varnothing}$.
(ii) Fix any $q' \geq 0$. Since $\mathrm{Re} g \leq |g| \leq \frac{3}{2}\mathrm{Re} g$ by hypothesis, we can conclude from [13, Proposition 3.9] that

$$\|e^{-g\tau_x^2}\|_{T_\phi} \leq e^{O(1+q'^2)|g|\mathfrak{h}^4} e^{-q'|g|\mathfrak{h}^4|\phi_x/\mathfrak{h}|^2}. \tag{5.8}$$

By the product property and (3.12), this gives

$$\|e^{-g\tau^2(B\setminus X)}\|_{T_\phi} \leq e^{O(1+q'^2)\epsilon_{g\tau^2}} e^{-q'|g|\mathfrak{h}^4 \sum_{x\in B\setminus X}|\phi_x/\mathfrak{h}|^2}. \tag{5.9}$$

For $Y \subset \Lambda$, we define the $L^2(Y)$ norm by

$$\|\phi\|_{L^2(Y)}^2 = \frac{1}{|Y|}\sum_{x\in Y}\frac{|\phi_x|^2}{\mathfrak{h}^2}. \tag{5.10}$$

Then we combine (5.9) with (5.7), using the product property, (1.81), and $|B \setminus X| \geq \frac{1}{2}|B|$, to obtain

$$\begin{aligned}\|\mathcal{I}(B\setminus X)\|_{T_\phi} &\leq e^{O(1+q'^2)\epsilon_{g\tau^2}} e^{-q'|g|\mathfrak{h}^4|B\setminus X|\,\|\phi\|_{L^2(B\setminus X)}^2} e^{\epsilon_V(1+\|\phi\|_\Phi^2)} \\ &\leq e^{O(1+q'^2)\epsilon_{g\tau^2}} e^{-\frac{1}{2}q'C_0^{-1}\epsilon_{g\tau^2}\,\|\phi\|_{L^2(B\setminus X)}^2} e^{\epsilon_V(1+\|\phi\|_\Phi^2)}\end{aligned} \tag{5.11}$$

(no L^d factor is produced for the observables). By our hypothesis on X and Proposition A.2,

$$\|\phi\|_{L^2(B\setminus X)}^2 \geq \frac{1}{2c_2^2}\|\phi\|_{\Phi(B^\square)}^2 - \|\phi\|_{\tilde{\Phi}(B^\square)}^2. \tag{5.12}$$

We insert this into (5.11) and localise the Φ norm to $\Phi(B^\square)$ to obtain

$$\|\mathcal{I}(B\setminus X)\|_{T_\phi} \leq e^{O(1+q'^2)\epsilon_V} e^{-(\frac{1}{4}C_0^{-1}c_2^{-2}q'\epsilon_{g\tau^2}-\epsilon_V)\,\|\phi\|_{\Phi(B^\square)}^2} e^{\frac{1}{2}q'\epsilon_V\|\phi\|_{\tilde{\Phi}(B^\square)}^2}. \tag{5.13}$$

Then (5.3) follows by choosing $q' = 4C_0c_2^2(q + C)$, which is $O(q)$. ∎

We prove Proposition 2.1 by combining Proposition 5.1 with the following elementary lemma.

Lemma 5.2. *For $x, u > 0$ and any integer $r \geq \max\{1, u\}$,*

$$(1 + x)^{2r} \leq (2r/u)^r e^{ux^2} \tag{5.14}$$

$$1 + u^r(1 + x)^{2r} \leq e^{2ru(1+x^2)}. \tag{5.15}$$

Proof. For the first bound, we combine $(1+x)^{2r} \le 2^r(1+x^2)^r$ with the inequality $1+x^2 \le (r/u)e^{ux^2/r}$ (since $r \ge u$). The second bound follows from

$$1+u^r(1+x)^{2r} \le 1+(2u)^r(1+x^2)^r \le (1+2u+2ux^2)^r \le (e^{2u+2ux^2})^r, \tag{5.16}$$

where we used $r \ge 1$ in the second inequality. ∎

Proof of Proposition 2.1. We first consider the choice $I^* = I(B)$. By the product property and [13, Proposition 3.10],

$$\begin{aligned} \|I(B)F\|_{T_\phi} &\le \|\mathcal{I}(B)\|_{T_\phi}\|1+W(B)\|_{T_\phi}\|F\|_{T_\phi} \\ &\le \|\mathcal{I}(B)\|_{T_\phi}\|1+W(B)\|_{T_\phi}\|F\|_{T_0}\left(1+\|\phi\|_\Phi\right)^r, \end{aligned} \tag{5.17}$$

where r denotes the degree of F. By (4.23), W is a degree-six polynomial in the boson and fermion fields. By (5.15) and [13, Proposition 3.10],

$$\|1+W(B)\|_{T_\phi} \le 1+\|W(B)\|_{T_\phi} \le 1+\|W(B)\|_{T_0}\left(1+\|\phi\|_\Phi\right)^6 \le e^{6\omega^{1/3}(1+\|\phi\|_\Phi^2)}. \tag{5.18}$$

where $\omega = \max_{B\in\mathcal{B}_j}\|W(B)\|_{T_0}$. Then, since $6(L^{2d}\omega)^{1/3} \le u$ by hypothesis, (5.14) gives

$$\|I(B)F\|_{T_\phi} \le \|\mathcal{I}(B)\|_{T_\phi}\|F\|_{T_0}\left(\frac{2r}{u}\right)^r e^{u+2u\|\phi\|_\Phi^2}. \tag{5.19}$$

Then (2.1) with $I^* = I(B)$ follows from (5.2). For (2.2), fix $q \ge 0$ to be the desired parameter in (2.2), and choose the variable called q in (5.3) to be q_1 defined by $q_1 = q + 2u\epsilon_{g\tau^2}^{-1}$. This gives (2.2) for the choice $I^* = I(B)$.

For the case $\tilde{I}(B\setminus X)$ with $X = \varnothing$ or $X \in \mathcal{S}_{j-1}$, we replace (5.18) by

$$\|\prod_{b\in\mathcal{B}_{j-1}(B\setminus X)}(1+W(b))\|_{T_\phi} \le e^{6L^d(L^{-d}\omega)^{1/3}(1+\|\phi\|_\Phi^2)} \le e^{u(1+\|\phi\|_\Phi^2)}, \tag{5.20}$$

and proceed similarly.

Omitting factors $1+W$ in the above bounds only makes it easier, so we also have the bounds if we choose I^* with factors of $1+W$ missing, and the proof is complete. ∎

Proof of Proposition 2.2. Let $V \in \bar{\mathcal{D}}$. We first consider the case $I_* = I$ (possibly with some $1+W$ factors omitted) and $j_* = j$. The bound (2.9) follows from (2.3) and (2.7) (with $q=0$), and (2.10) follows similarly from the case $r=0$. Also, for $B \in \mathcal{B}_j$, it follows from the definition of I, the product property, (1.79) and (4.2), that

$$\|I(B)^{-1}\|_{T_0} \le e^{\|V(B)\|_{T_0}}\frac{1}{1-\|W(V,B)\|_{T_0}} \le (1+O(\epsilon_V+\epsilon_W)) \le 2, \tag{5.21}$$

which gives (2.11). This completes the proof for the case $I_* = I$.

Next, we consider the case $I_* = \tilde{I}_{\rm pt}$. It follows from Proposition 1.4 that $V_{\rm pt} \in \bar{\mathcal{D}}'$, and the above result for $I_* = I$ then gives (2.9)–(2.11) also for $\tilde{I}_{\rm pt}$ when $j_* = j$.

This leaves (2.9)–(2.11) for the case $I_* = \tilde{I}_{\rm pt}$ with $j_* = j+1$, as well as (2.12). For (2.9), we apply Lemma 3.2 and the scale-j case of (2.9) (now W_{j+1} occurs rather than W_j but it is bounded by Remark 4.2) to obtain

$$\|\tilde{I}_{\rm pt}(B)F\|_{T_{\phi,j+1}(\mathfrak{h}_{j+1})} \leq \|\tilde{I}_{\rm pt}(B)F\|_{T_{\phi,j}(\mathfrak{h}_j)} \prec \|F\|_{T_{0,j}} \mathcal{G}_j(B,\phi), \tag{5.22}$$

where $\mathcal{G}_j = G_j$ for $\mathfrak{h}_j = \ell_j$, and $\mathcal{G}_j = \tilde{G}_j$ for $\mathfrak{h}_j = h_j$. For $\mathfrak{h} = \ell$ we set $\phi = 0$ and (2.9) immediately follows for $j+1$. For $\mathfrak{h} = h$ we use the fact that $\tilde{G}_j(X,\phi) \leq \tilde{G}^{\gamma}_{j+1}(X,\phi)$ by Lemma 1.2, and (2.9) also follows in this case. Note that $\|F\|_{T_{0,j}}$ occurs in (2.9) both for $j_* = j$ and $j_* = j+1$. The estimate (2.10) follows similarly, and (2.11) for $j+1$ follows from (2.11) for j by Lemma 3.2, which implies that the $T_{\phi,j+1}$ norm is bounded above by the $T_{\phi,j}$ norm.

Finally, to prove (2.12), we recall from Proposition 1.4 that $V_{\rm pt} \in \bar{\mathcal{D}}'_{j+1}$, and then (2.12) follows exactly as the scale-j case of (2.10) for $\tilde{I}_{\rm pt}$. This completes the proof. ∎

Proof of Proposition 2.3. We first prove the analyticity of $V \mapsto \mathcal{I} = e^{-V}$ for V in $\bar{\mathcal{D}}_j$; in this case $j_* = j$. We fix B and drop it from the notation.

Fix $V \in \bar{\mathcal{D}}_j$ and let $\dot{V} \in \mathcal{Q}$. We prove analyticity by showing that $I(V+\dot{V})$ has a norm convergent power series expansion in $\dot{V}$, if $|\dot{g}| \leq \frac{1}{8}{\rm Re}g$ and $\epsilon_{\dot{V}}$ is sufficiently small. By the integral form of the remainder in Taylor's theorem, together with the product property of the T_ϕ semi-norm,

$$\begin{aligned} \big\| e^{-(V+\dot{V})} - \sum_{n=0}^{N} e^{-V} \frac{(-\dot{V})^n}{n!} \big\|_j &= \big\| \int_0^1 \frac{1}{N!} e^{-(V+s\dot{V})} \dot{V}^{N+1} (1-s)^N ds \big\|_j \\ &\leq \sup_{\phi} \mathcal{G}(\phi)^{-1} \frac{1}{(N+1)!} \|e^{-V}\dot{V}^{N+1}\|_{T_\phi} e^{\|\dot{V}\|_{T_\phi}}, \end{aligned} \tag{5.23}$$

where $\mathcal{G}$ denotes the regulator, either G_j or $\tilde{G}_j$. It suffices to show that the above right-hand side goes to zero as $N \to \infty$, and for this it suffices to show that insertion of summation over N under the supremum leads to a convergent result. Since

$$\sum_{N=0}^{\infty} \frac{1}{(N+1)!} \|e^{-V}\dot{V}^{N+1}\|_{T_\phi} e^{\|\dot{V}\|_{T_\phi}} \leq \|e^{-V}\|_{T_\phi} e^{2\|\dot{V}\|_{T_\phi}}, \tag{5.24}$$

it suffices to show that

$$\sup_{\phi} \mathcal{G}(\phi)^{-1} \|e^{-V}\|_{T_\phi} e^{2\|\dot{V}\|_{T_\phi}} < \infty. \tag{5.25}$$

We isolate the τ^2 terms by writing $V = g\tau^2 + Q$ and $\dot{V} = \dot{g}\tau^2 + \dot{Q}$. By [13, Proposition 3.9], $\|\tau_x\|_{T_\phi} = \mathfrak{h}^2 P(t)$, where $P(t) = t^2 + 2t + 2$ and $t = |\phi_x|/\mathfrak{h}$. Let $\epsilon = \epsilon_V + 2\epsilon_{\dot{V}}$. We use the product property of the T_ϕ norm, as well as [13, Proposition 3.10], to obtain

$$\begin{aligned} \|e^{-V_x}\|_{T_\phi} e^{2\|\dot{V}_x\|_{T_\phi}} &\leq \|e^{-g\tau_x^2}\|_{T_\phi} e^{2|\dot{g}|\,\|\tau_x^2\|_{T_\phi} + \|Q_x\|_{T_\phi} + 2\|\dot{Q}_x\|_{T_\phi}} \\ &\leq \|e^{-g\tau_x^2}\|_{T_\phi} e^{2|\dot{g}|\mathfrak{h}^4 P(t)^2 + \epsilon L^{-dj}(1+\|\phi\|_\Phi^2)}. \end{aligned} \tag{5.26}$$

By [13, Proposition 3.9], together with the assumption in the definition of $\bar{\mathcal{D}}$ that $|{\rm Im}g| < \frac{1}{5}{\rm Re}g$,

$$\|e^{-g\tau_x^2}\|_{T_\phi} \leq e^{({\rm Re}g)\mathfrak{h}^4[-2t^2 + \frac{3}{2}P(t)^2]}. \tag{5.27}$$

Since $|\dot{g}| \leq \frac{1}{8}\mathrm{Re}g$, this gives

$$\|e^{-g\tau_x^2}\|_{T_\phi} e^{2|\dot{g}|\mathfrak{h}^4 P(t)^2} \leq e^{(\mathrm{Re}g)\mathfrak{h}^4[-2t^4+\frac{7}{4}P(t)^2]} \leq e^{(\mathrm{Re}g)\mathfrak{h}^4[q_1-q_2t^2]}, \tag{5.28}$$

where $q_2 \geq 0$ can be chosen arbitrarily with a corresponding choice of q_1. Therefore,

$$\|e^{-V_x}\|_{T_\phi} e^{2\|\dot{V}_x\|_{T_\phi}} \leq e^{(\mathrm{Re}g)\mathfrak{h}^4[q_1-q_2t^2]+\epsilon L^{-dj}(1+\|\phi\|_\Phi^2)}. \tag{5.29}$$

To conclude (5.25) for the G norm, we take $q_2 = 0$ and $\epsilon_{\dot{V}} = \epsilon_V$, and the desired estimate follows for uniformly small $\tilde{g}_j$. The proof of (5.25) for the $\tilde{G}$ norm can be completed by applying the Sobolev inequality exactly as in the proof of Proposition 5.1, using the fact that we do have $\epsilon_V \leq C\epsilon_{g\tau^2}$ in this case by (1.90).

It remains to consider the effect of $1+W$ on the above argument. Since $1+W$ is a degree-6 polynomial in the fields, it is analytic for the case of the G norm, and its effect is therefore unimportant. For the case of the $\tilde{G}$ norm, $1+W$ is not analytic because polynomial growth in the absolute value of ϕ is not cancelled by the regulator in this case (since the regulator has linear functions factored out). However, it is an exercise to include the factor $1+W$ alongside the e^{-V} factor in the above argument and thereby conclude analyticity also in this case.

To prove the analyticity of $\tilde{I}_{\mathrm{pt}}$ in $V \in \bar{\mathcal{D}}_j$, it again suffices to consider $e^{-V_{\mathrm{pt}}}$. Let $V \in \bar{\mathcal{D}}_j$ and consider first the case $j_* = j$. We can regard $e^{-V_{\mathrm{pt}}}$ as the composition of $V \mapsto V_{\mathrm{pt}}$ and $V_{\mathrm{pt}} \mapsto e^{-V_{\mathrm{pt}}}$. The first of these maps is polynomial in V. Thus, for the case of the G norm, $V \mapsto V_{\mathrm{pt}}$ is analytic, while the second map is analytic by the previous argument together with the fact that $V_{\mathrm{pt}} \in \bar{\mathcal{D}}'$ when $V \in \bar{\mathcal{D}}$ by Proposition 1.4. This proves the desired analyticity when $j_* = j$ for the G norm. The analyticity for the case of the $\tilde{G}$ norm can be established with small additional effort.

Next, we consider the case $j_* = j+1$. As above, the main work lies in showing that $e^{-V_{\mathrm{pt}}}$ is an analytic function of $V_{\mathrm{pt}} \in \bar{\mathcal{D}}$ when measured in the $\|\cdot\|_{j+1}$ norm. But it follows from Lemmas 3.2 and 1.2 that for either of the choices (1.58)–(1.59) for the norm pairs, $\|F\|_{j+1} \leq C\|F\|_j$ for some $C > 0$ and for all F. Thus convergence of a power series in a neighbourhood in the j-norm implies convergence in a neighbourhood in the $j+1$-norm, and the analyticity for $j_* = j+1$ follows from the analyticity for $j_* = j$.

Finally, it follows similarly that $I(B)^{-1}$ is analytic in V, as a map into the space with norm $\|\cdot\|_{T_{0,j}}$. For example, the factor $e^{g\tau^2(Y)}$ in $I(B)^{-1}$ is analytic in g because it has an absolutely convergent power series,

$$\|e^{g\tau^2(B)}\|_{T_0(\ell)} \leq \sum_{n\geq 0} \frac{1}{n!}\|g\tau^2(B)\|_{T_0(\ell)}^n \leq \sum_{n\geq 0} \frac{1}{n!}\epsilon_{g\tau^2}^n. \tag{5.30}$$

A similar argument applies to the inverse of $1-W$. This completes the proof. ∎

Proof of Proposition 2.4. Let $j < N$, $V \in \bar{\mathcal{D}}$, and $Q \in \mathcal{Q}$ with $\|Q(B)\|_{T_0} \prec \bar{\epsilon}$. We first show that $V - Q \in \bar{\mathcal{D}}'$. This implies that the estimates of Proposition 2.2 apply to $\hat{I}$, and that the desired analyticity follows from Proposition 2.3, so then it will remain only to prove the estimates (2.13)–(2.14).

By Lemma 3.1, $\epsilon_{V-Q} \leq \epsilon_V + \epsilon_Q \prec \epsilon_V + \max_B \|Q(B)\|_{T_0}$. The last bound of (3.8) (with worse constants) then follows from the assumption on Q. For the middle bound of (3.8), let g_Q denote

the coefficient of τ^2 in Q. By hypothesis, $L^{dj}|g_Q|\|\tau_0^2\|_{T_0(h)} \prec \tilde{g}_j^{1/4}$, and hence

$$L^{dj}|g-g_Q|\|\tau_0^2\|_{T_0(h)} \geq L^{dj}|g|\|\tau_0^2\|_{T_0(h)} - c_L\tilde{g}_j^{\eta} \geq ak_0^4 - c\tilde{g}_j^{1/4} \geq \frac{1}{2}ak_0^4, \tag{5.31}$$

by taking $\tilde{g}_j$ sufficiently small. Finally, for the first inequality of (3.8), we apply (3.11) to see that

$$|\mathrm{Im} g_Q| \leq |g_Q| \prec \frac{\epsilon_{Q,j}(\mathfrak{h}_j)}{L^{dj}\mathfrak{h}_j^4}. \tag{5.32}$$

By the hypothesis on Q, for $\mathfrak{h}=\ell$ the right-hand side is at most $c\ell_0^{-4}\tilde{g}_j$, which is at most $\frac{1}{10}C_{\mathcal{D}}^{-1}\tilde{g}_j < \frac{1}{10}\mathrm{Re} g$ for L sufficiently large (hence ℓ_0 large). Similarly, for $\mathfrak{h}=h$ the right-hand side is $\prec \tilde{g}_j^{5/4}$, and hence the effect of Q on the imaginary part of g is negligible. This completes the proof that $V-Q\in\bar{\mathcal{D}}'$.

It remains to prove (2.13)–(2.14). For $s\in[0,1]$, we write $V_s = V-sQ$, $I_s = I(V_s)$, $\mathcal{I}_s = e^{-V_s}$, and $W_s = W(V_s)$, and omit the B arguments. Direct calculation gives

$$I_s' = I_sQ + \mathcal{I}_sW_s', \tag{5.33}$$
$$I_s'' = I_sQ^2 + 2\mathcal{I}_sQW_s' + \mathcal{I}_sW_s'', \tag{5.34}$$
$$W_s' = -W(Q,V_s) - W(V_s,Q), \tag{5.35}$$
$$W_s'' = -2W(Q,Q). \tag{5.36}$$

By Lemma 4.12,

$$\|W_s'\|_{T_0} \prec_L \begin{cases} \chi_j\tilde{g}_j^2 & \mathfrak{h}=\ell \\ \chi_j\tilde{g}_j^{3/4} & \mathfrak{h}=h \end{cases} \qquad \|W_s''\|_{T_0} \prec_L \begin{cases} \chi_j\tilde{g}_j^2 & \mathfrak{h}=\ell \\ \chi_j\tilde{g}_j & \mathfrak{h}=h. \end{cases} \tag{5.37}$$

Let $\hat{I}(B) = I(V-Q,B)$. By the Fundamental Theorem of Calculus, $\hat{I}-I = \int_0^1 I_s'ds$, and hence by (5.33)

$$\|\hat{I}-I\|_j \leq \sup_{s\in[0,1]} \left(\|I_sQ\|_j + \|\mathcal{I}_sW_s'\|_j\right). \tag{5.38}$$

We have shown above that $V-sQ\in\bar{\mathcal{D}}'$ (in fact this holds uniformly in s), and consequently (2.9) holds with V replaced by $V-sQ$. By (2.9), the first term on the right-hand side of (5.38) is of order $\|Q\|_{T_0} \prec \bar{\epsilon}$. By (2.9) and (5.37), the second term of (5.38) is negligible compared to the first. This proves (2.13).

For (2.14), we first note that $I_1 - I_0 - I_0' = \hat{I} - I - IQ - \mathcal{I}_0W_0'$. Using this, with a second-order Taylor remainder estimate followed by (2.9), gives

$$\begin{aligned} \|\hat{I}-I-IQ\|_{T_0} &\leq \|\mathcal{I}_0W_0'\|_{T_0} + \sup_{s\in[0,1]}\|I_s''\|_{T_0} \\ &\prec \|W_0'\|_{T_0} + \|Q\|_{T_0}^2 + \sup_{s\in[0,1]}\left(\|Q\|_{T_0}\|W_s'\|_{T_0} + \|W_s''\|_{T_0}\right) \prec_L \bar{\epsilon}^2, \end{aligned} \tag{5.39}$$

where for the last step we used (5.37) together with the fact that its right-hand sides are at most $\bar{\epsilon}^2$. This proves (2.14). ∎

6 Proof of Propositions 2.5–2.6

In this section, we prove Propositions 2.5–2.6. The proof of Proposition 2.5 is short, whereas the proof of Proposition 2.6 is substantial. In the proof of Proposition 2.6 it is important that W and V_{pt} be defined as they are, and it is here that we implement the ideas in [6, Section 2].

6.1 Proof of Proposition 2.5

Proof of Proposition 2.5. Let $j < N$ and $V \in \bar{\mathcal{D}}_j$. Recall from (2.19) that $h_{\mathrm{lead}}(U, B)$ is defined for $(U, B) \in \mathcal{S}_{j+1} \times \mathcal{B}_{j+1}$ by

$$h_{\mathrm{lead}}(U,B) = \begin{cases} -\frac{1}{2}\mathbb{E}_{\pi,j+1}\theta(V_j(B); V_j(\Lambda \setminus B)) & U = B \\ \frac{1}{2}\mathbb{E}_{\pi,j+1}\theta(V_j(B); V_j(U \setminus B)) & U \supset B, |U|_{j+1} = 2 \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

By (2.17), (1.17) and (1.15),

$$\mathbb{E}_{\pi,C}(\theta A; \theta B) = F_{\pi,C}(e^{\mathcal{L}_C} A, e^{\mathcal{L}_C} B). \tag{6.2}$$

By Proposition 4.1,

$$\max_{B \in \mathcal{B}_{j+1}} \sum_{x \in B} \sum_{B' \in \mathcal{B}_{j+1}(\Lambda)} \|F_{\pi,C_{j+1}}(V_x, V(B'))\|_{T_{0,j+1}} \prec_L \bar{\epsilon}^2. \tag{6.3}$$

As an operator on the subspace of $\mathcal{N}$ consisting of bounded-degree polynomials in the fields, $e^{\pm \mathcal{L}_{C_k}}$ is bounded (uniformly in k), due to (4.21) and (1.72). With (4.1) and (6.2), this gives

$$\max_{B \in \mathcal{B}_{j+1}} \sum_{x \in B} \sum_{B' \in \mathcal{B}_{j+1}(\Lambda)} \|\mathbb{E}_{\pi,C_{j+1}}(\theta V_x; \theta V(B'))\|_{T_{0,j+1}} \prec_L \bar{\epsilon}^2, \tag{6.4}$$

from which we conclude that

$$\|h_{\mathrm{lead}}(U,B)\|_{T_0,j+1} \prec_L \bar{\epsilon}^2. \tag{6.5}$$

By Proposition 2.3, this implies that

$$\|\tilde{I}_{\mathrm{pt}}(U) h_{\mathrm{lead}}(U,B)\|_{j+1} \prec_L \bar{\epsilon}^2. \tag{6.6}$$

This gives (2.23) and completes the proof of Proposition 2.5. ∎

6.2 Proof of Proposition 2.6

We require some preparation for the proof of Proposition 2.6. By (2.19)–(2.20),

$$\begin{aligned} h_{\mathrm{lead}}(B) &= -\sum_{b \in \mathcal{B}_j(B)} \frac{1}{2}\mathbb{E}_\pi(\theta V(b); \theta V(\Lambda \setminus B)) \\ &= -\sum_{b \in \mathcal{B}_j(B)} \frac{1}{2}\mathbb{E}_\pi(\theta V(b); \theta V(\Lambda \setminus b)) + \sum_{b \neq b' \in \mathcal{B}_j(B)} \frac{1}{2}\mathbb{E}_\pi(\theta V(b); \theta V(b')). \end{aligned} \tag{6.7}$$

It follows from (2.18) that

$$\frac{1}{2}\mathbb{E}_\pi(V';V'')+\frac{1}{2}\mathbb{E}_\pi(V'';V')=\mathbb{E}(V';V''), \tag{6.8}$$

from which we conclude that

$$h_{\mathrm{lead}}(B)=-\sum_{b\in\mathcal{B}_j(B)}\frac{1}{2}\mathbb{E}_\pi(\theta V(b);\theta V(\Lambda\setminus b))+\sum_{b\neq b'\in\mathcal{B}_j(B)}\frac{1}{2}\mathbb{E}(\theta V(b);\theta V(b')). \tag{6.9}$$

For distinct $b,b'\in\mathcal{B}_j$, $B\in\mathcal{B}_{j+1}$, and for $U\in\mathcal{S}_{j+1}$ with $|U|_{j+1}\in\{1,2\}$, we define

$$R_1(b;B)=\tilde{I}_{\mathrm{pt}}^{B\setminus b}\mathbb{E}\delta I^b+\tilde{I}_{\mathrm{pt}}^{B}\frac{1}{2}\mathbb{E}_\pi(\theta V_j(b);\theta V_j(\Lambda\setminus b)), \tag{6.10}$$

$$R_2(b,b';U)=\frac{1}{2}\left[\tilde{I}_{\mathrm{pt}}^{U\setminus(b\cup b')}\mathbb{E}\delta I^{b\cup b'}-\tilde{I}_{\mathrm{pt}}^{U}\mathbb{E}(\theta V_j(b);\theta V_j(b'))\right]; \tag{6.11}$$

note that $\mathbb{E}_\pi$ appears in R_1 but not in R_2. Then, by (2.16), (2.19)–(2.20), and (6.9),

$$\tilde{I}_{\mathrm{pt}}^{B}[h_{\mathrm{red}}(B)-h_{\mathrm{lead}}(B)]=\sum_{b\in\mathcal{B}_j(B)}R_1(b;B)+\sum_{b\neq b'\in\mathcal{B}_j}R_2(b,b';B), \tag{6.12}$$

$$\tilde{I}_{\mathrm{pt}}^{U}[h_{\mathrm{red}}(U)-h_{\mathrm{lead}}(U)]=\sum_{b\neq b':\overline{b\cup b'}=U}R_2(b,b';U)\qquad\qquad |U|_{j+1}=2. \tag{6.13}$$

By the triangle inequality and (6.12)–(6.13), to prove Proposition 2.6 it suffices to show that

$$\|R_1(b;B)\|_{j+1}\prec_L\bar{\epsilon}^3,\qquad \|R_2(b,b';U)\|_{j+1}\prec_L\bar{\epsilon}^3, \tag{6.14}$$

where the constants in the upper bounds depend on L, and $\bar{\epsilon}$ is given by (1.91).

The appearance of δI leads naturally to the study of δV, which was defined in (3.19) as $\delta V=\theta V-V_{\mathrm{pt}}$. As a first step in the proof of (6.14), we prove the following lemma which relies heavily on results from [13]. The "5" appearing in its statement has been chosen as a convenient positive constant and is not significant. The parameter $\hat{\ell}_j>0$ is defined in (3.17). The constant $C_{\delta V}$ is the L-dependent constant of Lemma 3.4.

Lemma 6.1. *Let $j<N$, $b,b'\in\mathcal{B}_j$, and $n,n'\geq 0$. Suppose that $F\in\mathcal{N}((b\cup b')^\square)$ obeys $\|F\|_{T_\phi(\mathfrak{h}+\hat{\ell})}\leq c_F e^{\alpha\|\phi\|^2_{\Phi(\mathfrak{h})}}$ for some $c_F,\alpha>0$. If $u\in(0,2]$ obeys $\alpha+\frac{1}{20}(n+n')u\leq 5$, then*

$$\|\mathbb{E}_{j+1}\left[(\delta V(b))^n(\delta V(b'))^{n'}\theta F\right]\|_{T_\phi(\mathfrak{h})}\prec_L c_F(C_{\delta V}\bar{\epsilon})^{n+n'}e^{(2\alpha+(n+n')u)\|\phi\|^2_{\Phi(\mathfrak{h})}}, \tag{6.15}$$

where the constant in the upper bound depends on u,n,n', and where $\mathfrak{h}$, $\hat{\ell}$ and all norms are at scale $j+1$.

Proof. By [13, Proposition 3.19] (with (3.18) to provide its hypothesis on the covariance), and by the product property of the $T_{\phi\sqcup\xi}$ semi-norm,

$$\begin{aligned}\|\mathbb{E}\left[(\delta V(b))^n(\delta V(b'))^{n'}\theta F\right]\|_{T_\phi}&\leq\mathbb{E}\|(\delta V(b))^n(\delta V(b'))^{n'}\theta F\|_{T_{\phi\sqcup\xi}(\mathfrak{h}\sqcup\hat{\ell})}\\&\leq\mathbb{E}\left[\|\delta V(b)\|^n_{T_{\phi\sqcup\xi}(\mathfrak{h}\sqcup\hat{\ell})}\|\delta V(b')\|^{n'}_{T_{\phi\sqcup\xi}(\mathfrak{h}\sqcup\hat{\ell})}\|\theta F\|_{T_{\phi\sqcup\xi}(\mathfrak{h}\sqcup\hat{\ell})}\right].\end{aligned} \tag{6.16}$$

We apply [13, Proposition 3.10] to the $T_{\phi\sqcup\xi}(\mathfrak{h}\sqcup\hat{\ell})$ semi-norm of δV, with a multi-component field with $\mathfrak{h}=\hat{\ell}$ for ξ. With (3.20), this gives

$$\|\delta V(b)\|_{T_{\phi\sqcup\xi}(\mathfrak{h}\sqcup\hat{\ell})} \le \bar{\epsilon}(1+\|\phi\|_{\Phi(\mathfrak{h})})^4(1+\|\xi\|_{\Phi(\hat{\ell})})^4. \tag{6.17}$$

For any $u\in(0,2]$, (5.14) then gives (with a u-dependent constant and with $\hat{u}=u(\ell/\hat{\ell})^2$)

$$\|\delta V(b)\|_{T_{\phi\sqcup\xi}(\mathfrak{h}\sqcup\hat{\ell})} \prec C_{\delta V}\bar{\epsilon}e^{u(\|\phi\|^2_{\Phi(\mathfrak{h})}+\|\xi\|^2_{\Phi(\hat{\ell})})} = C_{\delta V}\bar{\epsilon}e^{u\|\phi\|^2_{\Phi(\mathfrak{h})}}G(b,\xi)^{\hat{u}}. \tag{6.18}$$

Similarly, by [13, Proposition 3.12], by hypothesis, by $\|\phi+\xi\|^2\le 2(\|\phi\|^2+\|\xi\|^2)$, and by $\mathfrak{h}\ge\ell$,

$$\begin{aligned}\|\theta F\|_{T_{\phi\sqcup\xi}(\mathfrak{h}\sqcup\hat{\ell})} &\le \|F\|_{T_{\phi+\xi}(\mathfrak{h}+\hat{\ell})} \le c_F e^{2\alpha(\|\phi\|^2_{\Phi(\mathfrak{h})}+\|\xi\|^2_{\Phi(\mathfrak{h})})}\\ &\le c_F e^{2\alpha\|\phi\|^2_{\Phi(\mathfrak{h})}}G(b\cup b',\xi)^{2\alpha}.\end{aligned} \tag{6.19}$$

Therefore, with $s=n+n'$, since $G(b\cup b')=G(b)G(b')$ by [13, (3.56)],

$$\|\mathbb{E}[(\delta V(b))^n(\delta V(b'))^{n'}\theta F]\|_{T_\phi} \prec (C_{\delta V}\bar{\epsilon})^s c_F e^{(2\alpha+su)\|\phi\|^2_{\Phi(\mathfrak{h})}}\mathbb{E}\left[G(b\cup b',\xi)^{2\alpha+s\hat{u}}\right]. \tag{6.20}$$

It suffices now to bound the expectation on the right-hand side by a constant. By (3.18), by our choice $\mathfrak{c}=\frac{1}{10}c_G$ above (1.73), and by (1.72) and (3.18),

$$\begin{aligned}(2\alpha+s\hat{u})\|C\|_{\Phi^+(\ell)} &= 2\alpha\|C\|_{\Phi^+(\ell)}+su\|C\|_{\Phi^+(\hat{\ell})}\\ &\le 2\alpha\mathfrak{c}+su\frac{c_G}{100} \le \Big(\frac{\alpha}{5}+\frac{su}{100}\Big)c_G \le c_G,\end{aligned} \tag{6.21}$$

with the last inequality true by hypothesis. Then [13, Proposition 3.20] yields the desired bound on the expectation, and the proof is complete. ∎

For $j\ge 1$, we define A_j by

$$A_j = e^{-\delta V}-\sum_{i=0}^{j-1}\frac{(-\delta V)^i}{i!}. \tag{6.22}$$

By Taylor's theorem with integral form of the remainder,

$$A_j = \frac{1}{(j-1)!}\int_0^1(1-t)^{j-1}(\delta V)^j e^{-t\delta V}dt. \tag{6.23}$$

It follows from the definitions that $e^{-\theta V}=e^{-V_{\rm pt}}e^{-\delta V}$, and that for $b\in\mathcal{B}_j$,

$$\delta I(b) = e^{-V_{\rm pt}(b)}\left(A_1(b)+Z(b)\right), \tag{6.24}$$

with

$$Z = e^{-\delta V}\theta W - W_{j+1}. \tag{6.25}$$

It is in the following proof that it is important that W and $V_{\rm pt}$ be defined as they are, and our implementation of the ideas laid out in [6, Section 2] occurs here. In particular, the identity

$$\mathbb{E}_{j+1}\theta W_j(V,X) = W_{j+1}(\mathbb{E}_{j+1}\theta V,X)+P(X)-\frac{1}{2}\mathbb{E}_{\pi,j+1}\big(\theta V(X);\theta V(\Lambda)\big) \tag{6.26}$$

of Lemma 4.5 enters the proof of (6.27) in a crucial manner, as does the definition $V_{\rm pt}=\mathbb{E}\theta V-P$ of (1.25) (recall (1.15)).

Proof of Proposition 2.6. All norms in this proof are at scale $j+1$. Fix $B \in \mathcal{B}_{j+1}$ and $b \in \mathcal{B}_j(B)$ for R_1, and fix $U \in \mathcal{S}_{j+1}$ with $|U| \in \{1,2\}$ and $b \neq b' \in \mathcal{B}_j$ with $\overline{b \cup b'} = U$ for R_2. To prove (6.14), it suffices to prove that

$$\|R_1(b;B)\|_{T_\phi} \prec_L \bar{\epsilon}^3 \mathcal{G}(B,\phi), \tag{6.27}$$

$$\|R_2(b,b';U)\|_{T_\phi} \prec_L \bar{\epsilon}^3 \mathcal{G}(U,\phi), \tag{6.28}$$

where $\mathcal{G}$ represents G or $\tilde{G}^\gamma$ according to the choice $\mathfrak{h} = \ell$ or $\mathfrak{h} = h$. We first prove the bound (6.27) for R_1, and then the bound (6.28) for R_2.

Identity for R_1. We apply (6.24), (6.22), and the definition $V_{\mathrm{pt}} = \mathbb{E}\theta V - P$, to obtain

$$\begin{aligned}
\delta I &= e^{-V_{\mathrm{pt}}}\left(-\delta V + A_2 + Z\right) \\
&= e^{-V_{\mathrm{pt}}}\left(-\delta V + \frac{1}{2}(\delta V)^2 + A_3 + (1+A_1)\theta W - W_{j+1}\right) \\
&= e^{-V_{\mathrm{pt}}}\left(\mathbb{E}\theta V - \theta V - P + \frac{1}{2}(\theta V - \mathbb{E}\theta V)^2 + \theta W - W_{j+1}(\mathbb{E}\theta V) + \mathcal{E}_1\right),
\end{aligned} \tag{6.29}$$

where

$$\mathcal{E}_1 = (\theta V - \mathbb{E}\theta V)P + \frac{1}{2}P^2 + A_3 + A_1\theta W + W_{j+1}(\mathbb{E}\theta V) - W_{j+1}(V_{\mathrm{pt}}). \tag{6.30}$$

Then, taking the expectation, we obtain

$$\mathbb{E}\delta I(b) = e^{-V_{\mathrm{pt}}}\left(-P + \frac{1}{2}\mathbb{E}(\theta V(b);\theta V(b)) + \mathbb{E}\theta W - W_{j+1}(\mathbb{E}\theta V) + \mathbb{E}\mathcal{E}_1\right), \tag{6.31}$$

with

$$\mathbb{E}\mathcal{E}_1 = \frac{1}{2}P^2 + \mathbb{E}A_3 + \mathbb{E}(A_1\theta W) + \big[W_{j+1}(\mathbb{E}\theta V) - W_{j+1}(V_{\mathrm{pt}})\big]. \tag{6.32}$$

It follows from (6.8) that $\mathbb{E}(\theta V(b);\theta V(b)) = \mathbb{E}_\pi(\theta V(b);\theta V(b))$. Thus, after application of (6.26), together with use of the identity

$$\mathbb{E}_\pi(\theta V(b);\theta V(b)) - \mathbb{E}_\pi(\theta V(b);\theta V(\Lambda)) = -\mathbb{E}_\pi(\theta V(b);\theta V(\Lambda \setminus b)), \tag{6.33}$$

we obtain

$$\begin{aligned}
\tilde{I}_{\mathrm{pt}}^{B\setminus b}\mathbb{E}\delta I(b) &= \tilde{I}_{\mathrm{pt}}^{B\setminus b}e^{-V_{\mathrm{pt}}(b)}\left(-\frac{1}{2}\mathbb{E}_\pi(\theta V(b);\theta V(\Lambda\setminus b)) + \mathbb{E}\mathcal{E}_1(b)\right) \\
&= -\tilde{I}_{\mathrm{pt}}^{B}\frac{1}{2}\mathbb{E}_\pi(\theta V(b);\theta V(\Lambda\setminus b)) \\
&\quad + \tilde{I}_{\mathrm{pt}}^{B\setminus b}e^{-V_{\mathrm{pt}}(b)}W_{j+1}\frac{1}{2}\mathbb{E}_\pi(\theta V(b);\theta V(\Lambda\setminus b)) + \tilde{I}_{\mathrm{pt}}^{B\setminus b}e^{-V_{\mathrm{pt}}(b)}\mathbb{E}\mathcal{E}_1(b).
\end{aligned} \tag{6.34}$$

By definition of R_1, this gives

$$R_1(b;B) = \tilde{I}_{\mathrm{pt}}^{B\setminus b}e^{-V_{\mathrm{pt}}(b)}W_{j+1}\frac{1}{2}\mathbb{E}_\pi(\theta V(b);\theta V(\Lambda\setminus b)) + \tilde{I}_{\mathrm{pt}}^{B\setminus b}e^{-V_{\mathrm{pt}}(b)}\mathbb{E}\mathcal{E}_1(b). \tag{6.35}$$

The use of (6.26) has led to an important cancellation which the definitions of W and V_{pt} were engineered to create.

Bound on R_1. It suffices to obtain a bound of the form $\bar{\epsilon}^3\mathcal{G}(B,\phi)$ for the T_ϕ semi-norms of each of the two terms on the right-hand side of (6.35), with the last of these terms given by (6.32). The resulting five terms are of two types: one type involves $\tilde{I}_{\rm pt}^{B\setminus b}e^{-V_{\rm pt}}$ multiplied by the polynomials $W_{j+1}\mathbb{E}_\pi(\theta V(b);\theta V(\Lambda\setminus b))$, P^2, $[W_{j+1}(\mathbb{E}\theta V)-W_{j+1}(V_{\rm pt})]$, and the second type involves two terms with expectations of the non-polynomial quantities A_1 and A_3.

For the first type of term, we apply (2.9) (the version with factor $(1+W(b))$ omitted) to conclude that, for a polynomial Q,

$$\|\tilde{I}_{\rm pt}^{B\setminus b}e^{-V_{\rm pt}(b)}Q(b)\|_{j+1} \prec \|Q(b)\|_{T_{0,j}}. \tag{6.36}$$

Bounds on the T_0 semi-norm of W_{j+1}, $\mathbb{E}_\pi(\theta V(b);\theta V(\Lambda\setminus b))$ and P follow from (4.2), (6.4), and (4.3). Also, the norm of $W_{j+1}(\mathbb{E}\theta V)-W_{j+1}(V_{\rm pt})$ is bounded in (4.58). With these bounds, and with epWbd, we obtain an upper bound of order $\bar{\epsilon}^3$ for the $(j+1)$-norm of the three terms with polynomials.

For the second type of term, we apply Lemma 6.1. For the A_3 term, it follows from (6.23) and the product property that

$$\|\tilde{I}_{\rm pt}^{B\setminus b}e^{-V_{\rm pt}(b)}\mathbb{E}A_3(b)\|_{T_\phi} \le \sup_{t\in[0,1]}\|\tilde{I}_{\rm pt}^{B\setminus b}e^{-(1-t)V_{\rm pt}(b)}\|_{T_\phi}\|\mathbb{E}\delta V(b)^3e^{-t\theta V(b)}\|_{T_\phi}. \tag{6.37}$$

By (3.6) and (2.1) (for its hypothesis on ω we see from (4.2) that $\omega \prec_L \bar{\epsilon}^2$), given any small $u_1>0$,

$$\|e^{-tV(b)}\|_{T_\phi(\mathfrak{h}+\hat{\ell})} \le \|e^{-tV(b)}\|_{T_\phi(2\mathfrak{h})} \le e^{O(\epsilon_V(2\mathfrak{h})+u_1)\|\phi\|^2_{\Phi(2\mathfrak{h})}} \le e^{O(\epsilon_V(\mathfrak{h})+u_1)\|\phi\|^2_{\Phi(\mathfrak{h})}}. \tag{6.38}$$

It therefore follows from Lemma 6.1 that given any small $u>0$, with a constant depending on u we have

$$\|\mathbb{E}\delta V(b)^3e^{-t\theta V(b)}\|_{T_\phi} \prec_L \bar{\epsilon}^3e^{O(\epsilon_V+u)\|\phi\|^2_\Phi}. \tag{6.39}$$

For the case of the regulator G, we bound the first factor on the right-hand side of (6.37) as follows. By (2.1), the product property, and (3.6),

$$\|\tilde{I}_{\rm pt}^{B\setminus b}e^{-(1-t)V_{\rm pt}(b)}\|_{T_{\phi,j+1}} \le \|\tilde{I}_{\rm pt}^{B\setminus b}\|_{T_{\phi,j+1}}\|e^{-(1-t)V_{\rm pt}(b)}\|_{T_{\phi,j}} \le e^{O(\epsilon_V+u)\|\phi\|^2_\Phi}. \tag{6.40}$$

Thus we obtain

$$\|\tilde{I}_{\rm pt}^{B\setminus b}e^{-V_{\rm pt}(b)}\mathbb{E}A_3(b)\|_{T_\phi} \prec_L \bar{\epsilon}^3G(B,\phi), \tag{6.41}$$

as required.

For the case of the regulator $\tilde{G}$, we take $u=u_1=\epsilon_{g\tau^2}$ and recall from (1.90) and (3.12) that $\epsilon_V \prec \epsilon_{g\tau^2} \asymp k_0^4$, with k_0 chosen small (recall the discussion above (2.7)). Then (6.39) gives, for some $c_0>0$,

$$\|\mathbb{E}\delta V(b)^3e^{-t\theta V(b)}\|_{T_\phi} \prec_L \bar{\epsilon}^3e^{c_0\epsilon_{g\tau^2}\|\phi\|^2_\Phi}. \tag{6.42}$$

We apply (2.2), with $q=c_0$, to see that

$$\|\tilde{I}_{\rm pt}^{B\setminus b}e^{-V_{\rm pt}(b)}\mathbb{E}A_3(b)\|_{T_\phi} \prec_L \bar{\epsilon}^3\tilde{G}^\gamma(B,\phi), \tag{6.43}$$

as required.

The $A_1\theta W_j$ term can be treated similarly, using Lemma 6.1 with $F = e^{-tV}W_j$. This completes the discussion of the bound on R_1.

Bound on R_2. Starting from the first line of (6.29), and recalling that Z is defined in (6.25), a little algebra leads to

$$\mathbb{E}\delta I^{b\cup b'} = e^{-V_{\rm pt}(b\cup b')}\big(\mathbb{E}(\theta V(b);\theta V(b')) + \mathcal{E}_2(b,b')\big), \tag{6.44}$$

where

$$\begin{aligned}\mathcal{E}_2(b,b') = P(b)P(b') - \mathbb{E}\big(\delta V(b)A_2(b')\big) - \mathbb{E}\big(A_2(b)\delta V(b')\big) + \mathbb{E}\big(A_2(b)A_2(b')\big)\\ + \mathbb{E}\big(A_1(b)Z(b')\big) + \mathbb{E}\big(Z(b)A_1(b')\big) + \mathbb{E}\big(Z(b)Z(b')\big).\end{aligned} \tag{6.45}$$

Therefore,

$$\begin{aligned}2R_2(b,b';U) = \tilde{I}_{\rm pt}^{U\setminus(b\cup b')}e^{-V_{\rm pt}(b\cup b')}\mathcal{E}_2(b,b')\\ + \tilde{I}_{\rm pt}^{U\setminus(b\cup b')}e^{-V_{\rm pt}(b\cup b')}[(1+W_{j+1}(b))(1+W_{j+1}(b'))-1]\mathbb{E}(\theta V(b);\theta V(b')).\end{aligned} \tag{6.46}$$

By (2.9) (with two missing $1+W$ factors), the $(j+1)$-norm of the second term on the right-hand side is bounded by a multiple of the T_0 semi-norm of the polynomial factor, which by (4.2) and (6.4) is of order $\bar{\epsilon}^4$. The contribution due to the PP term in $\mathcal{E}_2$ can be bounded in the same way, using (4.3). The six remaining terms in $\mathcal{E}_2$ can be handled in the same way as the A_3 and A_1 terms in $\mathcal{E}_1$, and we omit the details. Using Lemma 6.1, the $\delta V A_2$ and $A_1 Z$ terms are seen to be order $\bar{\epsilon}^3$, while the A_2A_2 and ZZ terms are order $\bar{\epsilon}^4$. In particular, it is not necessary to make use of any cancellation within Z. Together, these estimates produce an overall bound of order $\bar{\epsilon}^3$, and the proof is complete. ∎

7 Proof of Propositions 2.7–2.8

In this section, we prove Propositions 2.7–2.8.

7.1 Proof of Proposition 2.7

The main step in the proof of Proposition 2.7 is provided by the following lemma. The constant $C_{\delta L}$ is the L-dependent constant of Lemma 3.4.

Lemma 7.1. *Let $X, Y \in \mathcal{P}_j$ be disjoint. Let $F(Y) \in \mathcal{N}(Y^\square)$. There is an $\alpha_{\mathbb{E}} > 0$ (independent of L) and a $C_{\delta V} > 0$ (depending on L) such that*

$$\|\mathbb{E}\delta I^X \theta F(Y)\|_{T_\phi(\mathfrak{h}/2)} \le \alpha_{\mathbb{E}}^{|X|_j+|Y|_j}(C_{\delta V}\bar{\epsilon})^{|X|_j}\|F(Y)\|_{\mathcal{G}(\mathfrak{h})}\mathcal{G}(X\cup Y,\phi)^5, \tag{7.1}$$

where $\mathcal{G}$ denotes G or $\tilde{G}$ when $\mathfrak{h} = \ell$ or $\mathfrak{h} = h$, respectively. Norms and regulators are at scale j, the expectation represents $\mathbb{E}_{C_{j+1}}$, and δI is given by (2.15).

Proof. We write $\mathfrak{h}' = \mathfrak{h}/2$ and $\hat{\ell}' = \hat{\ell}/2$. By [13, Proposition 3.19] (with (3.18) to provide its hypothesis), and by the product property of the $T_{\phi\sqcup\xi}$ semi-norm,

$$\|\mathbb{E}\delta I^X \theta F(Y)\|_{T_\phi(\mathfrak{h}')} \leq \mathbb{E}\left[\|\delta I^X\|_{T_{\phi\sqcup\xi}(\mathfrak{h}'\sqcup\hat{\ell}')}\|\theta F(Y)\|_{T_{\phi\sqcup\xi}(\mathfrak{h}'\sqcup\hat{\ell}')}\right]. \tag{7.2}$$

By [13, Proposition 3.12] and (3.6) (with the fact that $\mathfrak{h} \geq \hat{\ell}$ for uniformly small $\tilde{g}_j$),

$$\|\theta F(Y)\|_{T_{\phi\sqcup\xi}(\mathfrak{h}'\sqcup\hat{\ell}')} \leq \|F(Y)\|_{T_{\phi+\xi}(\mathfrak{h}'+\hat{\ell}')} \leq \|F(Y)\|_{T_{\phi+\xi}(\mathfrak{h})} \leq \|F(Y)\|_{\mathcal{G}(\mathfrak{h})}\mathcal{G}(Y,\phi+\xi). \tag{7.3}$$

Since $\|\phi+\xi\|^2 \leq 2\|\phi\|^2 + 2\|\xi\|^2$, and since $\mathcal{G} \leq G$ because $\tilde{G} \leq G$, this gives

$$\|\theta F(Y)\|_{T_{\phi\sqcup\xi}(\mathfrak{h}'\sqcup\hat{\ell}')} \leq \|F(Y)\|_{\mathcal{G}(\mathfrak{h})}\mathcal{G}(Y,\phi)^2\mathcal{G}(Y,\xi)^2 \leq \|F(Y)\|_{\mathcal{G}(\mathfrak{h})}\mathcal{G}(Y,\phi)^2 G(Y,\xi)^2. \tag{7.4}$$

By (6.23)–(6.25), for $b \in \mathcal{B}_j$,

$$\begin{aligned}\|\delta I(b)\|_{T_{\phi\sqcup\xi}(\mathfrak{h}'\sqcup\hat{\ell}')} \leq & \|\delta V(b)\|_{T_{\phi\sqcup\xi}(\mathfrak{h}'\sqcup\hat{\ell}')} \sup_{t\in[0,1]} \|e^{-(1-t)V_{\rm pt}(b)}\|_{T_\phi(\mathfrak{h}')}\|\theta e^{-tV(b)}\|_{T_{\phi\sqcup\xi}(\mathfrak{h}'\sqcup\hat{\ell}')} \\ & + \|\theta(e^{-V(b)}W(b))\|_{T_{\phi\sqcup\xi}(\mathfrak{h}'\sqcup\hat{\ell}')} + \|e^{-V_{\rm pt}}W_{j+1}(b)\|_{T_\phi(\mathfrak{h}')}.\end{aligned} \tag{7.5}$$

By (6.18) (now interpreted at scale j rather than $j+1$; recall that the bound of Lemma 3.4 applies to either scale), for any choice of small positive u, and with $\hat{u} = u(\ell/\hat{\ell})^2$,

$$\|\delta V(b)\|_{T_{\phi\sqcup\xi}(\mathfrak{h}'\sqcup\hat{\ell}')} \prec C_{\delta V}\bar{\epsilon}e^{u\|\phi\|^2_{\Phi(\mathfrak{h}')}}G(b,\xi)^{\hat{u}}. \tag{7.6}$$

We now consider the supremum on the right-hand side of (7.5). Either $t \geq \frac{1}{2}$ or $1-t \geq \frac{1}{2}$. Suppose that $t \geq \frac{1}{2}$; the other case is simpler and we omit its details. By (2.10) and (3.6), $\|e^{-(1-t)V_{\rm pt}(b)}\|_{T_\phi(\mathfrak{h}')} \leq 2\mathcal{G}(b,\phi)$. By [13, Proposition 3.12], (3.6), the inequality $\|\phi\|^2 \leq 2\|\phi+\xi\|^2 + 2\|\xi\|^2$, and the identity $\|\phi\|_{\Phi(\mathfrak{h}')} = 2\|\phi\|_{\Phi(\mathfrak{h})}$,

$$\begin{aligned}\|\theta e^{-tV(b)}\|_{T_{\phi\sqcup\xi}(\mathfrak{h}'\sqcup\hat{\ell}')}e^{u\|\phi\|^2_{\Phi(\mathfrak{h}')}} &\leq \|e^{-tV(b)}\|_{T_{\phi+\xi}(\mathfrak{h}'+\hat{\ell}')}e^{u\|\phi\|^2_{\Phi(\mathfrak{h}')}} \\ &\leq \|e^{-tV(b)}\|_{T_{\phi+\xi}(\mathfrak{h})}e^{8u\|\phi+\xi\|^2_{\Phi(\mathfrak{h})}}e^{8u\|\xi\|^2_{\Phi(\mathfrak{h})}} \\ &\leq \|e^{-tV(b)}\|_{T_{\phi+\xi}(\mathfrak{h})}e^{8u\|\phi+\xi\|^2_{\Phi(\mathfrak{h})}}G(b,\xi)^{1/2},\end{aligned} \tag{7.7}$$

where we used $8u\|\xi\|^2_{\Phi(\mathfrak{h})} \leq \frac{1}{2}\|\xi\|^2_{\Phi(\ell)}$ in the last step (we can take $u \leq \frac{1}{16}$). Next, we apply (2.1) when $\mathcal{G} = G$, and (2.2) with $u = \epsilon_{g\tau^2}$ and $q = 8$ when $\mathcal{G} = \tilde{G}$, to obtain

$$\|e^{-tV(b)}\|_{T_{\phi+\xi}(\mathfrak{h})}e^{8u\|\phi+\xi\|^2_{\Phi(\mathfrak{h})}} \prec \mathcal{G}(b,\phi+\xi), \tag{7.8}$$

and hence

$$\begin{aligned}\|\theta e^{-tV(b)}\|_{T_{\phi\sqcup\xi}(\mathfrak{h}'\sqcup\hat{\ell}')}e^{u\|\phi\|^2_{\Phi(\mathfrak{h}')}} &\prec \mathcal{G}(b,\phi+\xi)G(b,\xi)^{1/2} \\ &\leq \mathcal{G}(b,\phi)^2\mathcal{G}(b,\xi)^2G(b,\xi)^{1/2}.\end{aligned} \tag{7.9}$$

Since $\mathcal{G} \leq G$, we conclude from the above estimates that

$$\begin{aligned}\|\delta V(b)\|_{T_{\phi\sqcup\xi}}\sup_{t\in[0,1]}\|e^{-(1-t)V_{\rm pt}(b)}\|_{T_\phi}\|\theta e^{-tV(b)}\|_{T_{\phi\sqcup\xi}} &\prec C_{\delta V}\bar{\epsilon}\mathcal{G}(b,\phi)^3G(b,\xi)^{\hat{u}+5/2} \\ &\prec C_{\delta V}\bar{\epsilon}\mathcal{G}(b,\phi)^3G(b,\xi)^{3(\ell/\hat{\ell})^2},\end{aligned} \tag{7.10}$$

using the fact that u is small and that $\hat{\ell} \le \ell$ by definition.

To complete the estimate on $\delta I(b)$, we now consider the two W terms in (7.5). By [13, Proposition 3.12], (3.6) and the fact that $\mathfrak{h} \ge \ell$, (2.9), and (4.2),

$$\begin{aligned} \|\theta(e^{-V(b)}W(b))\|_{T_{\phi\sqcup\xi}(\mathfrak{h}'\sqcup\hat{\ell}')} &\le \|e^{-V(b)}W(b)\|_{T_{\phi+\xi}(\mathfrak{h}'+\hat{\ell}')} \prec \|e^{-V(b)}W(b)\|_{T_{\phi+\xi}(\mathfrak{h})} \\ &\prec \|W(b)\|_{T_0}\mathcal{G}(b,\phi+\xi) \prec_L \bar{\epsilon}^2\mathcal{G}(b,\phi)^2\mathcal{G}(b,\xi)^2. \end{aligned} \tag{7.11}$$

Similarly (recall Remark 4.2),

$$\|e^{-V_{\mathrm{pt}}}W_{j+1}(b)\|_{T_\phi(\mathfrak{h}')} \prec \|e^{-V_{\mathrm{pt}}}W_{j+1}(b)\|_{T_\phi(\mathfrak{h})} \prec \|W_{j+1}(b)\|_{T_0(\mathfrak{h})}\mathcal{G}(b,\phi) \prec_L \bar{\epsilon}^2\mathcal{G}(b,\phi). \tag{7.12}$$

We are free to take $\bar{\epsilon}$ small depending on L, so that in the above two bounds $\prec_L \bar{\epsilon}^2$ can be replaced by a bound $\prec \bar{\epsilon}$.

The combination of (7.5) with (7.10)–(7.12) gives

$$\|\delta I(b)\|_{T_{\phi\sqcup\xi}(\mathfrak{h}\sqcup\ell)} \prec C_{\delta V}\bar{\epsilon}\mathcal{G}(b,\phi)^3 G(b,\xi)^{3(\ell/\hat{\ell})^2}. \tag{7.13}$$

As noted below Definition 1.1, $\mathcal{G}(X)\mathcal{G}(Y) = \mathcal{G}(X\cup Y)$. Thus there is a constant c (independent of L) such that

$$\|\delta I^X\|_{T_{\phi\sqcup\xi}} \le \prod_{b\in\mathcal{B}_j(X)} \|\delta I(b)\|_{T_{\phi\sqcup\xi}} \le (cC_{\delta V}\bar{\epsilon})^{|X|_j}\mathcal{G}(X,\phi)^3 G(X,\xi)^{3(\ell/\hat{\ell})^2}. \tag{7.14}$$

The proof is completed by inserting (7.4) and (7.14) into (7.2), also noting that

$$\mathbb{E}G(X\cup Y,\xi)^{3(\ell/\hat{\ell})^2} \le 2^{|X|_j+|Y|_j}. \tag{7.15}$$

This last inequality is a consequence of [13, Proposition 3.20], whose hypothesis is supplied by the fact that $3(\ell/\hat{\ell})^2\|C\|_{\Phi^+(\hat{\ell})} = 3\|C\|_{\Phi^+(\ell)} \le 3\mathfrak{c} \le c_G$ by our choice of $\mathfrak{c}$. ∎

Proof of Proposition 2.7. We apply Lemma 7.1 with scale-j norms and $\mathfrak{h} = \mathfrak{h}_j$. Since $\mathfrak{h}_{j+1} \le \mathfrak{h}' = \mathfrak{h}_j/2$, we can apply (3.6) to the left-hand side of (7.1) to conclude that

$$\|\mathbb{E}\delta I^X\theta F(Y)\|_{T_{\phi,j+1}(\mathfrak{h}_{j+1})} \le \alpha_{\mathbb{E}}^{|X|_j+|Y|_j}(C_{\delta V}\bar{\epsilon})^{|X|_j}\|F(Y)\|_{\mathcal{G}_j(\mathfrak{h}_j)}\mathcal{G}_j(X,\cup Y,\phi)^5. \tag{7.16}$$

For the norm pair (1.58), it suffices to consider the case $\phi = 0$, for which the regulator on the right-hand side of (7.16) reduces to unity and the integration property (2.25) immediately follows in this case. For the norm pair (1.59), Lemma 1.2 gives

$$\tilde{G}_j(X,\cup Y,\phi)^5 \le \tilde{G}_{j+1}^{\gamma}(X\cup Y,\phi), \tag{7.17}$$

and with (7.16) this gives (2.25) in this case. This completes the proof. ∎

7.2 Proof of Proposition 2.8

For convenience, we restate Proposition 2.8 as Proposition 7.2. Its proof uses Proposition 4.9 in a crucial way.

Proposition 7.2. *Let $j < N$ and $V \in \bar{\mathcal{D}}_j$. Let $X \in \mathcal{S}_j$ and $U = \overline{X}$. Let $F(X) \in \mathcal{N}(X^\square)$ be such that $\pi_\alpha F = 0$ when $X(\alpha) = \varnothing$. Then*

$$\|\tilde{I}_{\mathrm{pt}}^{U\setminus X}\mathbb{E}_{C_{j+1}}\theta F(X)\|_{j+1} \prec \gamma(X)\kappa_F + \kappa_{\mathrm{Loc}F}, \tag{7.18}$$

where $\kappa_F = \|F(X)\|_j$, $\kappa_{\mathrm{Loc}F} = \|\tilde{I}_{\mathrm{pt}}^X \mathrm{Loc}_X \tilde{I}_{\mathrm{pt}}^{-X} F(X)\|_j$, and where the pair of norms is given by either of (1.58) *or* (1.59).

Proof. We make the decomposition

$$F(X) = D(X) + E(X), \tag{7.19}$$

with

$$D(X) = \tilde{I}_{\mathrm{pt}}^X \mathrm{Loc}_X \tilde{I}_{\mathrm{pt}}^{-X} F(X), \qquad E(X) = \tilde{I}_{\mathrm{pt}}^X (1 - \mathrm{Loc}_X)\tilde{I}_{\mathrm{pt}}^{-X} F(X). \tag{7.20}$$

By the triangle inequality and the product property,

$$\|\tilde{I}_{\mathrm{pt}}^{U\setminus X}\mathbb{E}\theta F(X)\|_{j+1} \le \|\tilde{I}_{\mathrm{pt}}^{U\setminus X}\|_{j+1}\|\mathbb{E}\theta D(X)\|_{j+1} + \|\tilde{I}_{\mathrm{pt}}^{U\setminus X}\mathbb{E}\theta E(X)\|_{j+1}. \tag{7.21}$$

Since $X \in \mathcal{S}_j$, its closure U lies in $\mathcal{S}_{j+1}$ and hence consists of at most 2^d blocks. Therefore, by the product property and (2.12), $\|\tilde{I}_{\mathrm{pt}}^{U\setminus X}\|_{j+1} \le 2^{2^d}$. By the integration property of Proposition 2.7,

$$\|\mathbb{E}\theta D(X)\|_{j+1} \prec \|D(X)\|_j = \kappa_{\mathrm{Loc}F}. \tag{7.22}$$

Thus the D term in (7.19) leads to the final term of (7.18).

For the term involving E, we first apply the product property and [13, Proposition 3.19] (with its assumption given by $h \ge \ell$ and (1.72)) to obtain

$$\|\tilde{I}_{\mathrm{pt}}^{U\setminus X}\mathbb{E}\theta E(X)\|_{T_{\phi,j+1}(\mathfrak{h}_{j+1})} \le \|\tilde{I}_{\mathrm{pt}}^{U\setminus X}\|_{T_{\phi,j+1}(\mathfrak{h}_{j+1})}\mathbb{E}\|E(X)\|_{T_{\phi+\xi,j+1}(2\mathfrak{h}_{j+1})}. \tag{7.23}$$

We recall the inequality

$$\|F_1(1 - \mathrm{Loc}_X)F_2\|_{T'_\phi} \le \bar{C}\gamma(Y)\,(1 + \|\phi\|_{\Phi'})^{A+d+1} \sup_{0\le t\le 1}\left(\|F_1F_2\|_{T_{t\phi}} + \|F_1\|_{T_{t\phi}}\|F_2\|_{T_0}\right) \tag{7.24}$$

from Proposition 4.9 (where its notation is defined). To bound the semi-norm of $E(X)$, we apply (7.24) (writing $D = A + d + 1$ and $\gamma = \gamma(X)$), to obtain

$$\begin{aligned}\|E(X)\|_{T_{\phi+\xi,j+1}(2\mathfrak{h}_{j+1})} &\prec \gamma\left(1 + \|\phi + \xi\|_{\Phi_{j+1}(X^\square, 2\mathfrak{h}_{j+1})}\right)^D \\ &\times \sup_{0\le t\le 1}\left(\|F(X)\|_{T_{t(\phi+\xi),j}(\mathfrak{h}_j)} + \|\tilde{I}_{\mathrm{pt}}^X\|_{T_{t(\phi+\xi),j}(\mathfrak{h}_j)}\|\tilde{I}_{\mathrm{pt}}^{-X}\|_{T_{0,j}(\mathfrak{h}_j)}\|F(X)\|_{T_{0,j}(\mathfrak{h}_j)}\right).\end{aligned} \tag{7.25}$$

By the triangle inequality, the polynomial factor can be bounded as

$$\begin{aligned}\big(1+\|\phi+\|\xi\|_{\Phi_{j+1}(X^\square)}\big)^D &\le \big(1+\|\phi\|_{\Phi_{j+1}(X^\square)}\big)^D\big(1+\|\xi\|_{\Phi_{j+1}(X^\square)}\big)^D \\ &\prec \big(1+\|\phi\|_{\Phi_{j+1}(X^\square)}\big)^D G_{j+1}(X,\xi),\end{aligned} \tag{7.26}$$

where in the last step we used $\mathfrak{h}_{j+1} \ge \ell_{j+1}$ to conclude the inequality $\|\xi\|_{\Phi_{j+1}(2\mathfrak{h}_{j+1})} \le \|\xi\|_{\Phi_{j+1}(\ell_{j+1})}$, together with the fact that the regulator dominates polynomials by (5.14). Next, we apply (2.10)–(2.11) (the latter in conjunction with the product property), together with the definition of κ_F, to see that the quantity under the supremum in (7.25) is bounded above by a constant multiple of $\kappa_F \mathcal{G}_j(X,\phi+\xi)$. Using $\|\phi+\xi\|^2 \le 2\|\phi\|^2 + 2\|\xi\|^2$ to estimate this last regulator, we obtain

$$\begin{aligned}\|E(X)\|_{T_{\phi+\xi,j+1}(2\mathfrak{h}_{j+1})} &\prec \gamma\kappa_F\big(1+\|\phi\|_{\Phi_{j+1}(X^\square,2\mathfrak{h}_{j+1})}\big)^D \\ &\quad\times \mathcal{G}_j(X,\phi)^2\mathcal{G}_j(X,\xi)^2 G_{j+1}(X,\xi).\end{aligned} \tag{7.27}$$

Since $\mathcal{G} \le G$, we can then take the expectation using (1.73) (with Cauchy–Schwarz to separate the regulators at the two different scales), to obtain

$$\mathbb{E}\|E(X)\|_{T_{\phi+\xi,j+1}(2\mathfrak{h}_{j+1})} \prec \gamma\kappa_F\big(1+\|\phi\|_{\Phi_{j+1}(X^\square,2\mathfrak{h}_{j+1})}\big)^D \mathcal{G}_j(X,\phi)^2. \tag{7.28}$$

With (7.23), this gives

$$\begin{aligned}\|\tilde{I}_{\mathrm{pt}}^{U\setminus X}\mathbb{E}\theta E(X)\|_{T_{\phi,j+1}(\mathfrak{h}_{j+1})} &\prec \gamma\kappa_F\|\tilde{I}_{\mathrm{pt}}^{U\setminus X}\|_{T_{\phi,j+1}(\mathfrak{h}_{j+1})} \\ &\quad\times\big(1+\|\phi\|_{\Phi_{j+1}(X^\square,2\mathfrak{h}_{j+1})}\big)^D \mathcal{G}_j(X,\phi)^2.\end{aligned} \tag{7.29}$$

With an application of Proposition 2.1, this gives

$$\|\tilde{I}_{\mathrm{pt}}^{U\setminus X}\mathbb{E}\theta E(X)\|_{T_{\phi,j+1}(\mathfrak{h}_{j+1})} \prec \gamma\kappa_F\mathcal{G}_{j+1}(U,\phi)^{\gamma/2}\mathcal{G}_j(X,\phi)^2, \tag{7.30}$$

where the exponent $\gamma/2$ on $\mathcal{G}_{j+1}$ is a convenient choice.

For the norm pair (1.58) we set $\phi = 0$, the regulators become equal to 1, and the desired result follows from (7.30). For the norm pair (1.59), we apply Lemma 1.2 and $X \subset U$ to obtain

$$\tilde{G}_{j+1}(U,\phi)^{\gamma/2}\tilde{G}_j(X,\phi)^2 \prec \tilde{G}_{j+1}(U,\phi)^{\gamma/2}\tilde{G}_{j+1}(X,\phi)^{\gamma/2} \le \tilde{G}_{j+1}^{\gamma}(U,\phi), \tag{7.31}$$

and the desired result follows from (7.30). This completes the proof. ∎

A L^p norm estimates

Let $\phi : \Lambda \to \mathbb{C}$, and let $X \subset \Lambda$ be a subset of cardinality $|X|$. For $p \in [1,\infty)$, we define the L^p norm

$$\|\phi\|_{L^p(X)} = \frac{1}{\mathfrak{h}}\left(\frac{1}{|X|}\sum_{x\in X}|\phi(x)|^p\right)^{1/p}. \tag{A.1}$$

The weight $\mathfrak{h}$ is included in the norm so that, according to (1.32) and (1.36),

$$\|\phi\|_{L^p(X)}^p \le \|\phi\|_{\Phi(X)}^p. \tag{A.2}$$

Proposition A.2 below provides a lattice Sobolev inequality which shows that (A.2) can be reversed at the cost of an additional term. Our application of Proposition A.2 occurs in (5.12), with $p = 2$.

To prepare for the proposition, we first prove a lemma which shows that the reversal is possible for polynomials, even with an increase in the size of the domain of the Φ norm (recall that the small set neighbourhood $X^{\square}$ of X was defined in (1.37)). Throughout this appendix, we write $R = L^j$. The hypothesis below, that $R \geq R_0$, can then be achieved uniformly in j by taking L sufficiently large. Outside this appendix, we take the parameter p_Φ in the definition of the Φ norm to obey $p_\Phi \geq \frac{d+4}{2}$ (as mentioned in Section 1.1.6), but this restriction is unnecessary in the following lemma.

Lemma A.1. *Let $p_\Phi, q \geq 0$ be integers. Let Q denote the vector space of complex-valued polynomials defined on $\mathbb{R}^d$ and of degree at most q. Let f be the restriction of any polynomial in Q to $\mathbb{Z}^d$. Let B be a block of side R in $\mathbb{Z}^d$. There exists $c_0 = c_0(q,p) > 0$ such that for $R \geq R_0(q,p)$ sufficiently large,*

$$\|f\|_{\Phi(B^{\square})} \leq c_0 \|f\|_{L^p(B)}. \tag{A.3}$$

Proof. The inequality is homogeneous in $\mathfrak{h}$ so without loss of generality we take $\mathfrak{h} = 1$. It suffices to consider the case where $p_\Phi = q$. In fact, derivatives of $f \in Q$ having order higher than q vanish so the left-hand side of (A.3) is constant in $q \geq p_\Phi$, and the left-hand side is an increasing function of p_Φ so the statement is strongest when $p_\Phi = q$. Thus we take $p_\Phi = q$ throughout the proof. Moreover, (A.3) is trivial if f is a constant, so we consider the case $q \geq 1$.

Let $\mathcal{C}^q$ denote the space of q-times differentiable functions on $\mathbb{R}^d$ with norm given by

$$\|G\|_{\mathcal{C}^q} = \sup_{x \in \mathbb{R}^d} \sup_{|\alpha| \leq q} |D^\alpha G(x)|, \tag{A.4}$$

where α is a multi-index and D^α is the derivative on $\mathbb{R}^d$. Without loss of generality, we assume that B is centred at the origin of $\mathbb{Z}^d$. We obtain a continuum version $\hat{B}^{\square} \subset \mathbb{R}^d$ of $B^{\square}$ by placing a unit $\mathbb{R}^d$-cube centred at each point in $B^{\square}$. Let $I^{\square} = R^{-1}\hat{B}^{\square} \subset \mathbb{R}^d$ be its rescaled version. For $P \in Q$, let

$$\|P\|_{\mathcal{C}^q(I^{\square})} = \inf\{\|P - G\|_{\mathcal{C}^q} : G \in \mathcal{C}^q, G|_{I^{\square}} = 0\}. \tag{A.5}$$

This defines a norm on Q.

Given $F \in Q$, let f be the restriction of F to $\mathbb{Z}^d$, and let $\hat{F} \in Q$ be defined by $\hat{F}(x) = F(Rx)$ for $x \in \mathbb{R}^d$. We prove that

$$\|f\|_{\Phi(B^{\square})} \leq \|\hat{F}\|_{\mathcal{C}^q(I^{\square})}, \tag{A.6}$$

and that there is a $c_0(q,p) > 0$ and an $R_0(q,p)$ such that for $R \geq R_0$,

$$\|\hat{F}\|_{\mathcal{C}^q(I^{\square})} \leq c_0 \|f\|_{L^p(B)}. \tag{A.7}$$

Together, these two inequalities give (A.3).

We first prove (A.6). By Taylor's theorem, $R|\nabla^e f(x)| \leq \|D^e \hat{F}\|_{\mathcal{C}^0}$. By induction on $|\alpha|$, this gives

$$\sup_{x \in \mathbb{R}^d} |\nabla_R^\alpha f(x)| \leq \|\hat{F}\|_{\mathcal{C}^q}, \qquad |\alpha| \leq q, \tag{A.8}$$

where $\nabla_R^\alpha = R^{|\alpha|}\nabla^\alpha$. Given $\hat{G} \in \mathcal{C}^q$, let $g(x) = \hat{G}(R^{-1}x)$. By definition, $f(x) - g(x) = \hat{F}(R^{-1}x) - \hat{G}(R^{-1}x)$, so by (A.8) with $\hat{F}$ replaced by $\hat{F} - \hat{G}$,

$$\sup_{x \in \mathbb{Z}^d} |\nabla_R^\alpha [f(x) - g(x)]| \leq \|\hat{F} - \hat{G}\|_{\mathcal{C}^q}, \qquad |\alpha| \leq q. \tag{A.9}$$

Therefore,

$$\inf\left\{\|f-g\|_\Phi : \hat{G}\in\mathcal{C}^q, \hat{G}|_{I^\square}=0\right\} \le \|\hat{F}\|_{\mathcal{C}^q(I^\square)}. \tag{A.10}$$

The set of all lattice functions g with $g|_{I^\square}=0$ includes all functions g arising on the left-hand side, and the infimum over this larger class is smaller that the infimum in (A.10). Thus the left-hand side of (A.10) is greater than or equal to $\|f\|_{\Phi(B^\square)}$. This proves (A.6).

To prove (A.7), we define a second norm on Q, as follows. For $P\in Q$, let

$$\|P\|_{L^p(I)} = \left(\int_I |P(x)|^p dx\right)^{1/p}. \tag{A.11}$$

Since all norms on the finite-dimensional vector space Q are equivalent, there exists a constant $c_1=c_1(q,p)$ such that, for all $P\in Q$,

$$\|P\|^p_{\mathcal{C}^q(I^\square)} \le c_1 \|P\|^p_{L^p(I)}. \tag{A.12}$$

The difference

$$\|P\|^p_{L^p(I)} - \frac{1}{|B|}\sum_{x\in B}|P(R^{-1}x)|^p = \int_I |P(x)|^p dx - \frac{1}{|B|}\sum_{x\in B}|P(R^{-1}x)|^p \tag{A.13}$$

is a Riemann sum approximation error. It is therefore bounded in absolute value by R^{-1} times the maximum over I of $|DP^p|$, which is less than $pR^{-1}\|P\|^p_{\mathcal{C}^q(I^\square)}$ (here we use $q\ge 1$). Therefore,

$$\left(1-\frac{p}{R}c_1\right)\|P\|^p_{\mathcal{C}^q(I^\square)} \le c_1\frac{1}{|B|}\sum_{x\in B}|P(R^{-1}x)|^p. \tag{A.14}$$

We take R large enough that $1-\frac{p}{R}c_1\ge 1/2$, and set $P=\hat{F}$ in (A.14), to conclude (A.7) with $c_0=(2c_1)^{1/p}$. This completes the proof of (A.7), and hence of (A.3). ∎

Proposition A.2. *Let B be a block of side $R=L^j$ in the torus Λ of side length L^N, with $j\le N-1$. There are constants c_1, c_2 and R_0 (depending on p_Φ, p) such that for $X\subset B$ with $|X|\le c_1|B|$, and for $R\ge R_0$,*

$$\|\phi\|_{\Phi(B^\square)} \le c_2\left(\|\phi\|_{L^p(B\setminus X)} + \|\phi\|_{\tilde{\Phi}(B^\square)}\right). \tag{A.15}$$

Proof. The inequality (A.15) is homogeneous in $\mathfrak{h}$ so we may assume that $\mathfrak{h}=1$. For any $f\in\mathbb{C}^\Lambda$,

$$\|f\|^p_{L^p(B\setminus X)} \ge \frac{|B\setminus X|}{|B|}\|f\|^p_{L^p(B\setminus X)} = \|f\|^p_{L^p(B)} - \frac{|X|}{|B|}\|f\|^p_{L^p(X)}. \tag{A.16}$$

The restriction $j\le N-1$ is imposed to ensure that the periodicity of Λ plays no role, and we may assume that we are working on $\mathbb{Z}^d$ rather than on Λ. We apply Lemma A.1 with $q=1$. With c_0 the constant of Lemma A.1, let $c_1=(2c_0^p)^{-1}$. By hypothesis, $|X|\le(2c_0^p)^{-1}|B|$. Let $f\in Q$, with Q as in Lemma A.1. By (A.2) and the fact that $X\subset B^\square$, $\|f\|_{L^p(X)}\le\|f\|_{\Phi(X)}\le\|f\|_{\Phi(B^\square)}$. With Lemma A.1, this gives

$$\|f\|^p_{L^p(B\setminus X)} \ge \|f\|^p_{L^p(B)} - \frac{|X|}{|B|}\|f\|^p_{\Phi(B^\square)} \ge \left[\frac{1}{c_0^p}-\frac{1}{2c_0^p}\right]\|f\|^p_{\Phi(B^\square)}. \tag{A.17}$$

Therefore,

$$\|f\|_{\Phi(B^\square)} \le 2^{1/p} c_0 \|f\|_{L^p(B\setminus X)}. \tag{A.18}$$

Given $\phi : \mathbb{Z}^d \to \mathbb{C}$ and $f \in Q$, we apply the triangle inequality (twice), (A.18) and (A.2) to see that

$$\begin{aligned}
\|\phi\|_{\Phi(B^\square)} &\le \|f\|_{\Phi(B^\square)} + \|\phi - f\|_{\Phi(B^\square)} \\
&\le 2^{1/p} c_0 \|f\|_{L^p(B\setminus X)} + \|\phi - f\|_{\Phi(B^\square)} \\
&\le 2^{1/p} c_0 \|\phi\|_{L^p(B\setminus X)} + 2^{1/p} c_0 \|\phi - f\|_{L^p(B\setminus X)} + \|\phi - f\|_{\Phi(B^\square)} \\
&\le 2^{1/p} c_0 \|\phi\|_{L^p(B\setminus X)} + \big(2^{1/p} c_0 + 1\big) \|\phi - f\|_{\Phi(B^\square)}.
\end{aligned} \tag{A.19}$$

The desired inequality (A.15), with $c_2 = 2^{1/p} c_0 + 1$, then follows by minimising over $f \in Q$ once we note that

$$\inf\{\|\phi - f\|_{\Phi(B^\square)} : f \in V\} = \|\phi\|_{\tilde{\Phi}(B^\square)} \tag{A.20}$$

by definition of the norms in (1.36) and (1.39). ∎

B Further interaction estimates

This section comprises estimates of a more specialised nature, which are required in [15]. The estimates are stated as three lemmas. For the first lemma, for $B \in \mathcal{B}_j$ we define

$$\Delta I(B) = \tilde{I}(V, B) - I(V, B) = e^{-V(B)} \left[\prod_{b \in \mathcal{B}_{j-1}(B)} (1 + W_j(V, b)) - (1 + W_j(V, B)) \right]. \tag{B.1}$$

Lemma B.1. *For $j \le N$, for both choices of $\|\cdot\|_j$ in (1.58)–(1.59), for $B \in \mathcal{B}_j$ and $V \in \bar{\mathcal{D}}_j$,*

$$\|\Delta I(B)\|_j \prec_L \bar{\epsilon}^4. \tag{B.2}$$

Proof. By (B.1), together with the fact that $W_j(V, B) = \sum_{b \in \mathcal{B}_{j-1}(B)} W_j(V, b)$ by (1.20),

$$\Delta I(B) = e^{-V(B)} \sum_{\substack{X \in \mathcal{P}_{j-1}(B) : \\ |X|_{j-1} \ge 2}} \prod_{b \in \mathcal{B}_{j-1}(X)} W_j(V_j, b). \tag{B.3}$$

Then (4.2) gives a bound of order $(\bar{\epsilon}^2)^2$ for the T_0 semi-norm of the above sum, and the desired estimate follows from this together with (2.9). ∎

Lemma B.2. *For $V \in \bar{\mathcal{D}}$, $X \in \mathcal{S}$ and $F \in \mathcal{N}(X^\square)$,*

$$\Big\| \mathrm{Loc}_X \Big((I^{-X} - \tilde{I}^{-X}_{\mathrm{pt}}) F \Big) \Big\|_{T_0} \prec C_{\delta V} \bar{\epsilon} \|F\|_{T_0}. \tag{B.4}$$

All quantities and norms are at scale $j < N$, and norms are computed with either $\mathfrak{h} = \ell$ or $\mathfrak{h} = h$.

Proof. It follows from [14, Proposition 1.18] that

$$\Big\|\mathrm{Loc}_X\Big((I^{-X}-\tilde{I}_{\mathrm{pt}}^{-X})F\Big)\Big\|_{T_0} \prec \Big\|\Big((I^{-X}-\tilde{I}_{\mathrm{pt}}^{-X})F\Big)\Big\|_{T_0} \tag{B.5}$$

To estimate the right-hand side, we use the identity

$$\prod_i a_i^{-1}-\prod_i b_i^{-1}=\sum_k\Big(\prod_{i\le k}a_i^{-1}\Big)(a_k-b_k)\Big(\prod_{i\ge k}b_i^{-1}\Big), \tag{B.6}$$

the triangle inequality, the product property of the norm, and (2.11), to obtain

$$\Big\|\mathrm{Loc}_X\Big((I^{-X}-\tilde{I}_{\mathrm{pt}}^{-X})F\Big)\Big\|_{T_0} \prec \sup_{B\in\mathcal{B}(X)}\|I(B)-\tilde{I}_{\mathrm{pt}}(B)\|_{T_0}\|F\|_{T_0}. \tag{B.7}$$

We are thus reduced to estimates on a single block, and we henceforth omit B arguments.

To account for the fact that I involves W_j whereas $\tilde{I}_{\mathrm{pt}}$ involves W_{j+1}, we define $I_{\mathrm{pt}}=I(V_{\mathrm{pt}})=I_j(V_{\mathrm{pt}})$. Then

$$\|I-\tilde{I}_{\mathrm{pt}}\|_{T_0}\le\|I-I_{\mathrm{pt}}\|_{T_0}+\|I_{\mathrm{pt}}-\tilde{I}_{\mathrm{pt}}\|_{T_0}. \tag{B.8}$$

By (2.9) and (4.2), the second term on the right-hand side obeys

$$\|I_{\mathrm{pt}}-\tilde{I}_{\mathrm{pt}}\|_{T_0}=\|e^{-V_{\mathrm{pt}}}(W_j-W_{j+1})\|_{T_0}\prec_L\bar{\epsilon}^2. \tag{B.9}$$

To estimate the first term on the right-hand side of (B.8), we proceed as in the proof of (2.13) and now define $V_s=V+s(V_{\mathrm{pt}}-V)$, $I_s=I(V_s)$, $\mathcal{I}_s=e^{-V_s}$, and $W_s=W(V_s)$. The steps leading to (5.38) give

$$\|I-\tilde{I}_{\mathrm{pt}}\|_{T_0}\le\sup_{s\in[0,1]}\big(\|I_s\|_{T_0}\|(V_{\mathrm{pt}}-V)\|_{T_0}+\|\mathcal{I}_s\|_{T_0}\|W_s'\|_{T_0}\big). \tag{B.10}$$

The norms of I_s and $\mathcal{I}_s$ are bounded by 2, by (2.1). Also, $\|V_{\mathrm{pt}}-V\|_{T_0}$ was encountered in (3.24) and proved to be at most $C_{\delta V}\bar{\epsilon}$. With (4.59), we then obtain $\|W_s'\|_{T_0}\prec_l\bar{\epsilon}^2$. This completes the proof. ∎

The next lemma is applied in [15, Lemmas 6.1–6.2]. To prepare for its statement, given $V'\in\mathcal{Q}$ we define a new element $V''\in\mathcal{Q}$ by

$$V''=V'+y(\tau_\Delta-\tau_{\nabla\nabla}), \tag{B.11}$$

where y is the coefficient of $\tau_{\nabla\nabla}$ in V'. Thus the term $y\tau_{\nabla\nabla}$ in V' is replaced by $y\tau_\Delta$ to produce V''. We also define

$$\delta I^+(B)=e^{-V'(B)}\big(W_{j+1}(V',B)-W_{j+1}(V'',B)\big). \tag{B.12}$$

The definition of V'' is motivated by the fact that, for a polymer X, $V''(X)$ and $V'(X)$ are equal up to a polynomial in the fields that is supported on the boundary of X. To see this let χ, f and g be functions on Λ. Then

$$\begin{aligned}-\sum_{x\in\Lambda,e\in\mathcal{U}}(\nabla^e\chi)_x\big(\nabla^e(fg)\big)_x&=\sum_{x\in\Lambda}\chi_x\Big((\Delta f)_xg_x+f_x(\Delta g)_x\Big)\\&\quad+\sum_{x\in\Lambda,e\in\mathcal{U}}\chi_x(\nabla^ef)_x(\nabla^eg)_x.\end{aligned} \tag{B.13}$$

This is proved by using summation by parts (∇^e and ∇^{-e} are adjoints) to rewrite the summand on the left as $\chi_x(\Delta fg)_x$ followed by writing $\Delta(fg)_x$ as the sum over e in $\mathcal{U}$ of $f_{x+e}g_{x+e} - f_x g_x$ and using simple algebra. Choosing $f = \phi, \psi$ and $g = \bar{\phi}, \bar{\psi}$ and referring to (1.7) we obtain

$$- \sum_{x\in\Lambda, e\in\mathcal{U}} (\nabla^e\chi)_x(\nabla^e\tau)_x = 2\sum_{x\in\Lambda} \chi_x\Big(- \tau_{\Delta,x} + \tau_{\nabla\nabla,x}\Big). \tag{B.14}$$

For a polymer X in $\mathcal{P}_{j+1}$ let χ be the indicator function of X. Then from (B.11),

$$V''(X) - V'(X) = \frac{1}{2} y \sum_{x\in\Lambda, e\in\mathcal{U}} (\nabla^e\chi)_x(\nabla^e\tau)_x. \tag{B.15}$$

Let ∂X denote the points in X with a neighbour in $\Lambda \setminus X$. The right hand side is a sum of $\tau_{z'} - \tau_z$ over nearest neighbours z', z where z is in X and z' is not in X. By rewriting the fields in $\tau_{z'}$ using $f_{z'} = f_z + (\nabla^e f)_z$ we find that there exists a polynomial V_∂ which is quadratic in the fields and their derivatives such that

$$V''(X) - V'(X) = \sum_{z\in\partial X} V_{\partial,z} \tag{B.16}$$

and every term in $V_{\partial,z}$ has at least one derivative.

For $X \in \mathcal{P}_{j+1}$ and $B \in \mathcal{B}_{j+1}(\Lambda \setminus X)$, we set $R_X(B) = \delta I_X^{(6)}(B) = 0$ if B does not have a neighbour in ∂X, and otherwise define

$$R_X(B) = e^{-V_\partial(\partial X \cap B^1)} - 1, \qquad \delta I_X(B) = R_X(B) I(V'', B), \tag{B.17}$$

where $B^1 = B \cup \partial(\Lambda \setminus B)$.

Lemma B.3. *Let $j < N$, and $B \in \mathcal{B}_{j+1}$. Suppose that $V' \in \bar{\mathcal{D}}_{j+1}$ has $y\tau_{\nabla\nabla}$ term which obeys $\|y\tau_{\nabla\nabla}(b)\|_{T_{0,j}} \prec \bar{\epsilon}$ when $b \in \mathcal{B}_j$. Let $X \in \mathcal{P}_{j+1}$. Then for both choices of $\|\cdot\|_{j+1}$ in (1.58)–(1.59),*

$$\|\delta I^+(B)\|_{j+1} \prec_L \bar{\epsilon}^2, \tag{B.18}$$

$$\|\delta I_X(B)\|_{j+1} \prec \bar{\epsilon}. \tag{B.19}$$

Proof. By direct calculation, $\|\tau_{\nabla\nabla}(b)\|_{T_{0,j}} \asymp L^{(d-2)(j)}\mathfrak{h}_j^2$, and the right-hand side is ℓ_0^2 for $\mathfrak{h} = \ell$ and $k_0^2\tilde{g}_j^{-1/2}$ for $\mathfrak{h} = h$. Therefore, by hypothesis and by definition of $\bar{\epsilon}$, we have

$$|y| \prec \begin{cases} \ell_0^{-2}\bar{\epsilon} & \mathfrak{h} = \ell \\ k_0^{-2}\tilde{g}_j^{1/2}\bar{\epsilon} & \mathfrak{h} = h. \end{cases} \tag{B.20}$$

Since V_∂ is given by a sum over $O(L^{(d-1)(j+1)})$ boundary points of terms containing at least one gradient and two fields, this gives

$$\begin{aligned} \|V_\partial(\partial X \cap B)\|_{T_0} &\prec \begin{cases} \ell_0^{-2}\bar{\epsilon}L^{(d-1)(j+1)}L^{-(j+1)}\ell_{j+1}^2 & \mathfrak{h} = \ell \\ k_0^{-2}\tilde{g}^{1/2}\bar{\epsilon}L^{(d-1)(j+1)}L^{-(j+1)}h_{j+1}^2 & \mathfrak{h} = h \end{cases} \\ &= \bar{\epsilon}. \end{aligned} \tag{B.21}$$

To prove (B.18), we apply (2.9) to obtain

$$\|\delta I^+(B)\|_{j+1} \prec \|W_{j+1}(V', B) - W_{j+1}(V'', B)\|_{T_0(\mathfrak{h})}, \tag{B.22}$$

and then use (4.59)–(4.60) to see that the right-hand side is $\prec_l \bar{\epsilon}^2$ as required. (In fact we use a small variation of (4.59)–(4.60) in which we regard $V' - V''$ as supported on $\partial X \cap B$, with (B.21).)

To prove (B.19), we set $I_\partial(t) = I(V'', B)e^{-tV_\partial}$, with $V_\partial = V_\partial(\partial X \cap B)$. By the Fundamental Theorem of Calculus,

$$\delta I_X(B) = V_\partial \int_0^1 I_\partial(t)dt, \tag{B.23}$$

and hence

$$\|\delta I_X(B)\|_{j+1} \leq \sup_{t\in[0,1]} \|I_\partial(t)V_\partial\|_{j+1}. \tag{B.24}$$

The polynomial V'' obeys our stability estimates since compared to V' its $z\tau_\Delta$ term is modified by $z \mapsto z + y$ and this change is such that $\epsilon_{V''} \leq \epsilon_{V'}$, and hence $V'' \in \bar{\mathcal{D}}$. By [13, (5.23)] and [13, Proposition 3.10], $\|e^{-tV_\partial}\|_{T_\phi} \leq e^{\|V_\partial\|_{T_\phi}} \leq e^{\|V_\partial\|_{T_0}(1+\|\phi\|_\Phi^2)}$. The bound on $\|e^{-tV_\partial}\|_{T_\phi}$ is no larger than the effect of Q handled in (5.7), and thus e^{-tV_∂} is a negligible perturbation of $I(V'', B)$, and $I_\partial(t)$ also obeys the stability bounds. Thus we obtain from (2.9) and (B.21) that

$$\|\delta I_X(B)\|_{j+1} \prec \|V_\partial\|_{T_0} \prec \bar{\epsilon}, \tag{B.25}$$

and the proof is complete. ∎

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