This assignment is due at the beginning of class on Friday, March 11.

1. In the two-parameter Ehrenfest urn, $M$ balls are divided between urn 1 and urn 2. At each step, a ball is selected at random. If it was selected from urn 1 then it is placed in urn 2 with probability $t$ and otherwise it is returned to urn 1. If it was selected from urn 2 then it is placed in urn 1 with probability $s$ and otherwise it is returned to urn 1. Assume that $st \neq 0$.
   (a) What are the transition probabilities for this Markov chain?
   (b) Explain why this Markov chain has a unique stationary distribution.
   (c) Determine the stationary distribution.
   (d) This Markov chain is reversible even when $s \neq t$. Does that surprise you? Explain in a few sentences how the detailed balance is achieved.

2. The Bernoulli–Laplace model of diffusion is defined as follows. There are $M$ balls in each of urn 1 and urn 2. A total of $b$ balls are black (with $b \leq M$) and the remaining $2M - b$ balls are white. At each time, we pick one ball from each urn and interchange them. Let $X_n$ denote the number of black balls in urn 1 at time $n$. Hint: read about the hypergeometric distribution.
   (a) Explain why this Markov chain has a unique stationary distribution.
   (b) What are the transition probabilities?
   (c) Determine the stationary distribution.

3. Recall that a finite graph is a set $G = (V,E)$ where $V$ is an arbitrary finite set whose elements are called vertices, and $E$ is a set of unordered pairs of vertices with each pair in $E$ called an edge. Let $|E|$ denote the number of edges in $G$. Let $d_j$ denote the degree of vertex $j$ (number of neighbours connected to $j$ by an edge). A particle performs random walk on the vertex set of a connected graph $G$. At each stage, it moves along an edge to a neighbour of its current position, with each such neighbour chosen with equal probability.
   (a) Explain why this Markov chain has a unique stationary distribution.
   (b) Let $\pi_j = \frac{d_j}{2|E|}$, for $j \in V$. Show that $\pi$ obeys $\pi = \pi P$ and $\sum_{j \in V} \pi_j = 1$.
   (c) Is the Markov chain reversible? Prove or disprove.

4. A Markov chain with state space $S = \{0, 1, 2, \ldots\}$ has transition probabilities $p_{i,i+1} = a_i$ and $p_{i,0} = 1 - a_i$ where each $a_i$ obeys $0 < a_i < 1$. Let $b_0 = 1$, $b_i = a_0 a_1 \cdots a_i$ for $i \geq 1$.
   (a) Show that the chain is recurrent if and only if $\lim_{i \to \infty} b_i = 0$.
   (b) Show that the chain is positive recurrent if and only if $\sum_{i=0}^{\infty} b_i < \infty$.
   (c) Fix $A > 0$ and $\beta > 0$, and suppose that $a_i = 1 - Ai^{-\beta}$ for all sufficiently large $i$.
      i. Show that the chain is transient if $\beta > 1$.
      ii. Show that the chain is positive recurrent if $\beta < 1$.
      iii. Now suppose $\beta = 1$ and show that the chain is positive recurrent if $A > 1$ and null recurrent if $A \leq 1$.

5. Smith has four umbrellas (total) at home and at the casino. When travelling between home and casino, he takes an umbrella if it is raining, if there is one at the location where he departs. If it is not raining, he does not take an umbrella. It rains each trip independently with probability $p$.
   (a) Set this up as a Markov chain, where $X_n$ is the number of umbrellas at his current location.
(b) Find the stationary distribution of this Markov chain.
(c) What is the probability that Smith gets wet (because it is raining and he has no umbrella with him)?
(d) What value of $p$ is the worst climate for Smith, i.e., where he gets wet most often?

The following problems from Durrett are recommended for extra practice but are not to be handed in: 6.5.4, 6.5.5, 6.5.6, 6.6.1.