Math 419/545 Assignment 6  February 26, 2016  Dr. G. Slade

This assignment is due at the beginning of class on Friday, March 4.

1. (a) For the gambler’s ruin problem with total fortune \(N\), let \(M_i\) denote the expected number of games that will be played when Smith initially has \(i\) \((i = 0, 1, \ldots, N)\). Let \(q = 1 - p\). Show that
\[
M_0 = M_N = 0, \quad M_i = 1 + p M_{i+1} + q M_{i-1} \quad (i = 1, \ldots, N - 1).
\]
(b) Solve the equations in (a) to obtain
\[
M_i = i(N - i) \quad \text{if} \quad p = \frac{1}{2},
\]
\[
M_i = \frac{i}{q - p} - \frac{N - (q/p)^i}{q - p 1 - (q/p)^N} \quad \text{if} \quad p \neq \frac{1}{2},
\]
by proceeding as follows. First, find the general solution to the homogeneous equation \(M_i = p M_{i+1} + q M_{i-1}\) (already done in class for \(p \neq \frac{1}{2}\), you will need to do it for \(p = \frac{1}{2}\)). Next, find a particular solution to the inhomogeneous equation \(M_i = 1 + p M_{i+1} + q M_{i-1}\) (try \(M_i = ci^2\) for \(p = \frac{1}{2}\); find \(c\) that produces a solution). Add the general solution of the homogeneous equation to the particular solution of the inhomogeneous equation. Finally, solve for the two unknown constants in the general solution by using the boundary conditions.

(c) Determine \(M_i\) for the case of infinite resources \((N = \infty)\).

2. Find the intercommunicating classes, and determine which are closed, recurrent, transient, periodic (with which period), for the Markov Chain with state space \(\{1, 2, 3, 4, 5, 6\}\) and transition matrix
\[
\begin{pmatrix}
\frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

3. Let \(p \in [0, 1]\). Consider the Markov Chain \((X_n)_{n \geq 0}\) with state space \(\{1, 2, 3\}\) and transition matrix
\[
P = \begin{pmatrix}
0 & 1 & 0 \\
1 - p & 0 & p \\
0 & 1 & 0
\end{pmatrix}.
\]
(a) Verify that \(P^2 = P^4\).
(b) Find \(P^n\) for all \(n \geq 1\).
(c) Suppose that the initial distribution is \(\mu^{(0)} = (\frac{1}{3}, \frac{2}{3}, 0)\). Determine \(\mu^{(n)}_j = P(X_n = j)\) for \(j = 1, 2, 3\) and \(n \geq 1\).
(d) Suppose again that \(\mu^{(0)} = (\frac{1}{3}, \frac{2}{3}, 0)\). What can you say about the limit of \(P(X_n = j)\) as \(n \to \infty\)?

4. (a) Durrett 6.3.5. (In the notation used in class, \(\tau_C = H_C\).)
(b) Durrett 6.3.7.
(c) Durrett 6.3.8.

The following problems from Durrett are recommended for extra practice but are not to be handed in: 6.3.3, 6.3.4, 6.3.5, 6.3.11, 6.4.2, 6.4.3, 6.4.4.