This assignment is due at the beginning of class on Wednesday, January 17.

1. Suppose that $A_1, A_2, \ldots$ is a partition of $\Omega$, i.e., $\bigcup_{i=1}^\infty A_i = \Omega$ and $A_i \cap A_j = \emptyset \forall i \neq j$. Suppose that $P(A_i) > 0$ for all $i$. Let $\mathcal{G}$ be the $\sigma$-algebra generated by the sets $A_i$. It was shown in class that the conditional expectation $E(X \mid \mathcal{G})$ is given by

$$E(X \mid \mathcal{G})(\omega) = \frac{E(X 1_{A_i})}{P(A_i)} \quad \text{for } \omega \in A_i.$$

Suppose now that $X, Y$ are discrete random variables, and define $E(X \mid Y)$ to be equal to $E(X \mid \mathcal{G})$, where $\mathcal{G} = \sigma(Y)$ is the $\sigma$-algebra determined by $Y$ (i.e., the smallest $\sigma$-algebra containing $\{Y \leq y\}$ for all real $y$). Let $p(x, y)$ be the joint probability mass function of $(X, Y)$. Show that

$$Z = \frac{\sum_x x p(x, Y)}{\sum_x p(x, Y)}$$

is a version of $E(X \mid Y)$ and that this recovers the elementary definition of conditional expectation, namely $E(X \mid Y = y) = \sum_x x p(x, y)/p(Y = y)$. (If the denominator is zero then set $Z = 0$; this happens with probability zero.)

2. Conditional probability is defined by $P(A \mid \mathcal{G}) = E(1_A \mid \mathcal{G})$ and $P(A \mid B) = P(A \cap B)/P(B)$ for events $A, B$ with $P(B) \neq 0$.

(a) For $\mathcal{G} = \sigma(A_1, A_2, \ldots)$ as in #1, show that $P(A \mid \mathcal{G})$ is the random variable that takes the value $P(A \mid A_i)$ when $A_i$ occurs. In particular, if $\mathcal{G} = \{\emptyset, B, B^c, \Omega\}$ then $P(A \mid \mathcal{G})$ is $P(A \mid B)$ when $B$ occurs and is $P(A \mid B^c)$ when $B$ does not occur.

(b) For $A \in \mathcal{G}$ (arbitrary $\mathcal{G}$), prove Bayes’ Formula:

$$P(A \mid B) = \frac{E(1_A P(B \mid \mathcal{G}))}{E(P(B \mid \mathcal{G}))}.$$

(c) For $\mathcal{G} = \sigma(A_1, A_2, \ldots)$ as in #1, show that Bayes’ Formula becomes

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{\sum_j P(B \mid A_j)P(A_j)}.$$

3. For random variables $W, Z$ with $E|Z| < \infty$, we define $E(Z \mid W) = E(Z \mid \sigma(W))$. Suppose $X, Y$ are random variables with $EX^2 < \infty$ and $EY^2 < \infty$. Suppose that $E(X \mid Y) = Y$ and $E(Y \mid X) = X$. Prove that $X = Y$ a.s.

Hint: Consider $E(X - Y)^2$.

4. Let $\xi_1, \xi_2, \ldots$ be independent with $E\xi_i = 0$ and $\text{Var} \xi_i = \sigma_i^2 < \infty$. Let $S_n = \sum_{i=1}^n \xi_i$ and $s_n^2 = \sum_{i=1}^n \sigma_i^2$. Let $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$. Show that $S_n^2 - s_n^2$ is a martingale.

5. One gambling strategy is to play one game $X$ until some stopping time $N$ and then to switch to a different game $Y$. At the time of switching, the gambler’s capital is $X_N$ and he or she may choose to carry forward all or part of it in the next game. So suppose that $X_n$ and $Y_n$ are supermartingales with respect to a filtration $\mathcal{F}_n$, and let $N$ be a stopping time such that $X_N \geq Y_N$ (when $N < \infty$). Show that

$$Z_n = X_n 1_{n < N} + Y_n 1_{n \geq N}$$

is a supermartingale. Since $N + 1$ is a stopping time,

$$Z'_n = X_n 1_{n \leq N} + Y_n 1_{n > N}$$

is also a supermartingale. In the latter case, the first game is played up to and including time $N$, and then the gambler switches to the second game.

The following problems from Durrett are recommended for extra practice but are not to be handed in: 5.1.3, 5.1.4, 5.1.8, 5.1.9, 5.1.10, 5.2.1, 5.2.2.