Problems to hand in:

1. Let $X, X_1, X_2, \ldots$ be random variables on the same probability space. Let $p \geq 1$. We say that $X_n \to X$ in $L^p$ (or in $p$th mean) if $\lim_{n \to \infty} \|X_n - X\|_p = 0$.

   (a) Show that if $X_n \to X$ in $L^p$ then $X_n \to X$ in probability.
   
   (b) Give a counterexample to the converse of (a).
   
   (c) Let $q \geq p \geq 1$. Show that if $X_n \to X$ in $L^q$ then $X_n \to X$ in $L^p$.

2. (a) Suppose that the sequence of measures $\{\mu_n\}$ is tight. Show that their characteristic functions are equicontinuous (i.e., for all $\epsilon > 0$ there exists $\delta > 0$ such that if $|h| < \delta$ then $|\phi_n(t + h) - \phi_n(t)| < \epsilon$ for all $n$).

   (b) Suppose that $\mu_n \Rightarrow \mu$. Use (a) to conclude that $\phi_n(t) \to \phi(t)$ uniformly on compact sets.

   (c) Give an example to show that the convergence in (b) need not be uniform on the entire real line.

3. Using characteristic functions, prove the following:

   (a) Suppose $X_i$ are independent with $N(0, \sigma_i^2)$ distributions. Let $S_n = X_1 + \cdots + X_n$. Then $S_n$ has distribution $N(0, \sum_{i=1}^n \sigma_i^2)$. In particular, if $Z_i$ has a standard normal $N(0, 1)$ distribution then $\frac{1}{\sqrt{n}}(Z_1 + \cdots + Z_n)$ also has a standard normal distribution.

   (b) Suppose $X_i$ are independent Cauchy random variables (density $f(x) = \frac{1}{\pi (1+x^2)}$, $x \in \mathbb{R}$). Let $S_n = X_1 + \cdots + X_n$. Then $\frac{1}{n}S_n$ has a Cauchy distribution.

   (c) Recall that a Geometric($p$) random variable has p.m.f. $g(k) = (1-p)^{k-1}p$ for $k \in \mathbb{N}$. Consider the following variations: $X_n$ has p.m.f. $P(X_n = k/n) = (1-\lambda/n)^{k-1}(\lambda/n)$ (typo corrected from earlier version) for $k \in \mathbb{N}$, with $\lambda > 0$. Then $X_n$ converges weakly to an Exp($\lambda$) random variable.

4. This problem concerns the method of Monte Carlo integration, which is a method for the approximate evaluation of an integral $I = \int_0^1 f(x)dx$.

   (a) Let $U_1, \ldots, U_N$ be i.i.d. uniform random variables on the interval $(0, 1)$, and let

   $$ I_N = \frac{1}{N}[f(U_1) + \cdots + f(U_N)]. $$

   Suppose that $\int_0^1 f(x)^2 dx < \infty$, and let $\sigma^2 = \text{Var}(U_1) = \int_0^1 f(x)^2 dx - I^2$. Apply the central limit theorem to show that $I_N$ converges to $I$ as $N \to \infty$, in the sense that

   $$ P \left( \left| I_N - I \right| \leq \frac{\sigma x}{\sqrt{N}} \right) \to P(|Z| \leq x), $$

   where $Z$ is a standard normal random variable.

   (b) Assuming that $\sigma \leq 1$, how large should $N$ be taken to be 95% confident that $I_N$ is within 0.01 of $I$? For this you will need a table of the c.d.f. $\Phi$ of a standard normal random variable, e.g., https://en.wikipedia.org/wiki/Standard_normal_table.

Problems not to be handed in:

11.5.2, 11.5.4, 11.5.6, 11.5.12.

For solutions, see: http://www.probability.ca/jeff/grprobbook.html.