1. An ecology graduate student goes to a pond containing \( n \) water beetles and captures 60 of them, marks each with a dot of paint, and then releases them. A few days later she goes back and captures another sample of 50. Let \( X \) denote the number of marked beetles found in her sample of 50.

(a) Determine the probability mass function of \( X \), i.e., find \( P(X = k) \) for each \( k \).

(b) She finds 12 beetles in her sample. Show that the function \( P(X = 12) \) is initially an increasing function of \( n \) which then becomes decreasing after reaching a maximum value. Find the maximum likelihood estimate for \( n \); that is the value of \( n \) which maximizes \( P(X = 12) \).

2. A searchlight is distance 1 from an infinitely long wall. Let \( Q \) denote the closest point on the wall and assume the searchlight scans along the wall so that at any given time, the angle \( \Theta \) the beam of light makes is uniform on \((−\pi/2, \pi/2)\) (with the point \( Q \) corresponding to angle 0). Let \( X \in \mathbb{R} \) (positive or negative) be the position of the beam on the wall as measured from \( Q \). Find the distribution and density functions for \( X \).

3. Let \( X_1, X_2, \ldots \) be independent and identically distributed \( \text{Exp}(1) \) random variables. Thus \( P(X_i > x) = e^{-x} \) for \( x \geq 0 \).

(a) Show that \( \lim \sup_{n \to \infty} X_n / \log n = 1 \) a.s.

(b) Let \( M_n = \max_{1 \leq m \leq n} X_m \). Show that \( M_n / \log n \to 1 \) a.s.

4. For \( d \in \mathbb{N} \), the integer lattice \( \mathbb{Z}^d \) is the set of all \( d \)-component vectors of integers. A bond is a pair \( \{x,y\} \) of adjacent elements of \( \mathbb{Z}^d \), i.e., with \( \|x - y\|_1 = \sum_{i=1}^{d} |x_i - y_i| = 1 \). In bond percolation, bonds are independently occupied with probability \( p \) and vacant with probability \( 1 - p \), where \( p \in [0,1] \) is fixed.

It is straightforward to modify the construction of the probability space for coin flips (Section 2.6) to model a biased coin (heads with probability \( p \) and tails with probability \( 1 - p \) for fixed \( p \in [0,1] \)), by setting \( P(A_{a_1a_2\cdots a_n}) = p^h(1-p)^{n-h} \), where \( h = \sum_{i=1}^{n} a_i \) denotes the number of heads among \( a_1a_2\cdots a_n \).

(a) Let \( E \) be the event that there is an infinite connected cluster of occupied bonds. Prove that \( E \) truly is an event, i.e., is in the \( \sigma \)-algebra. Prove that \( E \) is in fact a tail event for the sequence of events \( O_1, O_2, \ldots \), where \( O_n \) is the event that the \( n^{th} \) bond is occupied.

(b) Let \( E_0 \) be the event that the origin is in an infinite cluster. Prove or disprove: \( E_0 \) is a tail event for the sequence of events \( O_1, O_2, \ldots \).

Recommended problems. The following problems are recommended for extra practice but are not to be handed in:

6.3.2, 6.3.4, 6.3.6.

For solutions, see: http://www.probability.ca/jeff/grprobbook.html.