This assignment is due at the beginning of class on Friday, October 4.

1. An ecology graduate student goes to a pond containing $n$ water beetles and captures 60 of them, marks each with a dot of paint, and then releases them. A few days later she goes back and captures another sample of 50. Let $X$ denote the number of marked beetles found in her sample of 50.

(a) Determine the probability mass function of $X$, i.e., find $P(X = k)$ for each $k$.

(b) She finds 12 beetles in her sample. Show that the function $P(X = 12)$ is initially an increasing function of $n$ which then becomes decreasing after reaching a maximum value. Find the maximum likelihood estimate for $n$; that is the value of $n$ which maximizes $P(X = 12)$.

2. A searchlight is distance 1 from an infinitely long wall. Let $Q$ denote the closest point on the wall and assume the searchlight scans along the wall so that at any given time, the angle $\Theta$ the beam of light makes is uniform on $(-\pi/2, \pi/2)$ (with the point $Q$ corresponding to angle 0). Let $X \in \mathbb{R}$ (positive or negative) be the position of the beam on the wall as measured from $Q$. Find the distribution and density functions for $X$.

3. Let $X_1, X_2, \ldots$ be independent and identically distributed Exp(1) random variables. Thus $P(X_i > x) = e^{-x}$ for $x \geq 0$.

(a) Show that $\lim \sup_{n \to \infty} X_n / \log n = 1$ a.s.

(b) Let $M_n = \max_{1 \leq m \leq n} X_m$. Show that $M_n / \log n \to 1$ a.s.

4. For $d \in \mathbb{N}$, the integer lattice $\mathbb{Z}^d$ is the set of all $d$-component vectors of integers. A bond is a pair $\{x, y\}$ of adjacent elements of $\mathbb{Z}^d$, i.e., with $\|x - y\|_1 = \sum_{i=1}^d |x_i - y_i| = 1$. In bond percolation, bonds are independently occupied with probability $p$ and vacant with probability $1 - p$, where $p \in [0, 1]$ is fixed.

It is straightforward to modify the construction of the probability space for coin flips (Section 2.6) to model a biased coin (heads with probability $p$ and tails with probability $1 - p$ for fixed $p \in [0, 1]$), by setting $P(A_{a_1a_2\cdots a_n}) = p^h(1-p)^{n-h}$, where $h = \sum_{i=1}^n a_i$ denotes the number of heads among $a_1a_2\cdots a_n$.

(a) Let $E$ be the event that there is an infinite connected cluster of occupied bonds. Prove that $E$ truly is an event, i.e., is in the $\sigma$-algebra. Prove that $E$ is in fact a tail event.

(b) Let $E_0$ be the event that the origin is in an infinite cluster. Prove or disprove: $E_0$ is a tail event.

A bond percolation cluster on (part of) $\mathbb{Z}^2$, with $p = 0.500000$, from http://vbeffara.perso.math.cnrs.fr/.


Recommended problems. The following problems are recommended for extra practice but are not to be handed in:

6.3.2, 6.3.4, 6.3.6.

For solutions, see: http://www.probability.ca/jeff/grprobbook.html.