Math 418/544 Assignment 1

September 13, 2019

Dr. G. Slade

This assignment is due at the beginning of class on Friday, September 20.

1. This problem provides an example illustrating why it is not always possible to assign a probability to every subset of the sample space, i.e., why we introduce a σ-field $\mathcal{F}$. In this problem, the sample space is $\Omega = \{1, 2, 3, \ldots\}$, the set of natural numbers. Given a subset $F \subset \Omega$, let $N_n(F)$ denote the number of natural numbers up to and including $n$ which belong to the set $F$. We investigate the possibility of interpreting the limiting frequency

$$
\lim_{n \to \infty} \frac{N_n(F)}{n}
$$

as the probability of $F$.

(a) Let $A$ denote the set of natural numbers which are divisible by 3. Show that the above limit exists for $F = A$ and equals $\frac{1}{3}$. It is natural to interpret this as saying that a randomly chosen natural number will be divisible by 3 with probability $P(A) = \frac{1}{3}$.

(b) Let $E$ be a finite subset of $\Omega$. Show that the limiting frequency $\lim_{n \to \infty} n^{-1}N_n(E)$ is zero. Conclude that if we attempt to define $P(F) = \lim_{n \to \infty} n^{-1}N_n(F)$ for each subset $F$ of $\Omega$, then $P$ will not be a probability measure. Define a σ-field on which $P$ is a well-defined probability measure (answer here is not unique).

2. Let $S = \{1, 2, \ldots, n\}$ and suppose that $A$ and $B$ are each equally likely to be any of the $2^n$ subsets of $S$. Show that

$$
P(A \subset B) = \left(\frac{3}{4}\right)^n.
$$

3. 2.7.5

4. 2.7.7

5. Let $\mathcal{F}$ be the σ-field generated by an arbitrary nonempty collection of sets $\{E_\alpha : \alpha \in A\}$. Prove that for each $E \in \mathcal{F}$, there exists a countable subcollection $\{E_{\alpha_j} : j = 1, 2, \ldots\}$ (depending on $E$) such that $E$ already belongs to the σ-field generated by this subcollection. Hint: Consider the class of all sets with the asserted property and show that it is a σ-field containing each $E_\alpha$.

Recommended problems. The following problems from Rosenthal are recommended for extra practice but are not to be handed in:

2.7.2, 2.7.4, 2.7.6, 2.7.14, 2.7.22.

For solutions, see: http://www.probability.ca/jeff/grprobbook.html.