Math 321 Assignment 8: Due Friday, March 20 at start of class

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Midterm 2 will be held in class on Wednesday, March 25. It covers the material on Assignments 5–8; this corresponds to the text p. 151 (Theorem 7.16) to p. 185 (Theorem 8.8).

1. Prove that \( \log(1 + x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \) for \( |x| < 1 \). The simplest way to do this is to integrate the geometric series.
   (You may find this useful in the next problem.)

2. Prove the following:
   (a) If \( a > 0 \) then \( \lim_{x \to 0} \frac{a^x - 1}{x} = \log a \).
   (b) \( \lim_{x \to 0} \frac{1}{x} \left( \log(1 + x) - x \right) = -\frac{1}{2} \).
   (c) If \( x \in \mathbb{R} \) then \( \lim_{n \to \infty} (1 + x/n)^n = e^x \).

3. Suppose that \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) has radius of convergence \( R > 0 \), and that \( f(0) \neq 0 \). Prove that it is possible to expand \( 1/f(x) \) in a power series in an interval \((-r, r)\) for some \( r > 0 \).
   Hint: write \( f(x) = f(0) - (f(0) - f(x)) \) and use a geometric series.

4. For \( t \in \mathbb{R} \), we define \( f_t : \mathbb{R} \to \mathbb{R} \) by
   \[
   f_t(x) = \begin{cases} 
   \frac{e^{tx}}{e^t - 1} & (x \neq 0) \\
   1 & (x = 0).
   \end{cases}
   \]
   (a) Show that \( f_t \) can be written as a power series about \( x = 0 \), which converges in some interval \((-r, r)\).
   (b) For \( |x| < r \), we define \( P_n(t) \) \( (n = 0, 1, 2, \ldots) \) by
   \[
   f_t(x) = \sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!}.
   \]
   Explain why \( \sum_{n=0}^{\infty} P_n(t) \frac{x^n}{n!} = e^{tx} \sum_{n=0}^{\infty} P_n(0) \frac{x^n}{n!} \) and use this to conclude that
   \[
   P_n(t) = \sum_{k=0}^{n} \binom{n}{k} P_k(0) t^{n-k}.
   \]
   In particular, each \( P_n \) is a polynomial of degree \( n \). These polynomials are called the Bernoulli polynomials, and \( B_n = P_n(0) \) are called the Bernoulli numbers.
   (c) Prove that:
   i. \( B_0 = 1, B_1 = -\frac{1}{2}, \) and \( \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \) for \( n \geq 2 \).
   ii. \( P_n'(t) = nP_{n-1}(t) \) for \( n \in \mathbb{N} \).
   iii. \( P_n(t+1) = P_n(t) + t^{n-1} \) for \( n \in \mathbb{N} \).

Practice problems (not to be handed in):
Chapter 8: #5, 6, 7, 8, 9.
The solutions manual is here: http://digital.library.wisc.edu/1793/67009.
Prove that: \( \lim_{x \to -\infty} \log x = -\infty \), \( \lim_{x \to \infty} \log x = \infty \).
Continuation of #4. Prove that:
(i) \( P_n(1-t) = (-1)^n P_n(t) \) for \( n \geq 0 \), (ii) \( B_{2n+1} = 0 \) for \( n \in \mathbb{N} \),
(iii) \( \sum_{m=1}^{k-1} m^n = \frac{P_{n+1}(k) - P_{n+1}(0)}{n+1} \) for \( k, n = 2, 3, \ldots \).