Math 321 Assignment 7: Due Friday, March 9 at start of class

Dr. G. Slade

You must staple your pages together when you submit your assignments.

1. (a) Let \( f : [1, \infty) \to \mathbb{R} \) be continuous and suppose that \( \lim_{x \to \infty} f(x) \) exists. Let \( \epsilon > 0 \). Prove that there is a polynomial \( P \) such that
\[
\sup_{x \in [1, \infty)} |f(x) - P(1/x)| < \epsilon.
\]
(b) Give an example of a continuous function \( f : [1, \infty) \to \mathbb{R} \) which cannot be uniformly approximated by a polynomial in \( 1/x \).

2. Suppose that \( R \in [0, \infty] \) is the radius of convergence of the power series \( \sum_{n=0}^{\infty} a_n x^n \) and that \( a_n \neq 0 \) for all \( n \). Prove that
\[
\lim_{n \to \infty} \frac{|a_n|}{a_{n+1}} \leq R \leq \limsup_{n \to \infty} \frac{|a_n|}{a_{n+1}}.
\]

3. This problem presents an alternate approach to Theorem 8.3. Do not appeal to Theorem 8.3 in your proof — you are developing a different proof.

(a) For \( i, j \in \mathbb{N} \) let
\[
a_{ij} = \begin{cases} 
1 & (j = i) \\
-1 & (j = i + 1) \\
0 & \text{otherwise.}
\end{cases}
\]
Compute each of \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \) and \( \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} \).

(b) Suppose that \( a_{ij} \geq 0 \) for all \( i, j \). Prove that \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \) and \( \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} \) are equal to each other, and are equal to
\[
S = \sup \left\{ \sum_{i,j \in I} a_{i,j} : I \text{ is a finite subset of } \mathbb{N} \times \mathbb{N} \right\}.
\]
(It is possible that \( S = \infty \) in which case both double sums are also infinite.)

(c) Suppose that \( \sum_{i=1}^{\infty} (\sum_{j=1}^{\infty} |a_{ij}|) < \infty \). Prove that \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} \).
Hint: you may find it useful to write \( a_{ij} = a_{ij}^+ - a_{ij}^- \) where \( a_{ij}^+ = \max(a_{ij}, 0) \) and \( a_{ij}^- = \max(-a_{ij}, 0) \).

4. Let \( p \) be a rational number which is not a nonnegative integer. (This problem also can be applied for \( p \in \mathbb{R} \) once we have proved that \( \frac{d}{dy} y^p = py^{p-1} \) for real \( p \) as in Rudin (8.44).)

(a) Let \( \left( \frac{p}{n} \right) = 1 \) and for \( n \in \mathbb{N} \) let \( \left( \frac{p}{n} \right) = \frac{1}{n!} \prod_{k=0}^{n-1} (p-k) \). Prove that the radius of convergence of the series \( \sum_{n=0}^{\infty} \left( \frac{p}{n} \right) x^n \) is equal to 1. (What happens when \( p \) is a nonnegative integer?)

(b) For \( |x| < 1 \), let \( g(x) = \sum_{n=0}^{\infty} \left( \frac{p}{n} \right) x^n \). Prove that \( (1+x)g(x) = pg(x) \) with \( g(0) = 1 \), and conclude that \( g(x) = (1+x)^p \) for \( |x| < 1 \).

5. If each \( a_n \geq 0 \), if \( \sum_{n=1}^{\infty} a_n = \infty \), and if \( \sum_{n=0}^{\infty} a_n x^n \) converges for \( |x| < 1 \), prove that \( \lim_{x \to 1^-} \sum_{n=0}^{\infty} a_n x^n = \infty \).

Practice problems (not to be handed in):
Chapter 8: #1, 2, 3, 11.