Math 321 Assignment 5: Due Friday, February 28 at start of class

Please read instructions for assignment submission at:

1. Read the definition of a rectifiable curve in Definition 6.26. Prove that the curve \( f : [0,1] \rightarrow \mathbb{R} \) defined by Theorem 7.18 (the continuous nowhere differentiable function) is not rectifiable. In other words, it has infinite length.
   Hint: the proof of Theorem 7.18 holds the key.

2. Prove that the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin(1 + \frac{x}{n}) \) converges uniformly on \([-R,R]\) for every \( R > 0 \).
   Hint: One possibility is to employ the alternating series test https://en.wikipedia.org/wiki/Alternating_series_test.

3. Suppose that \( f_n : [a,b] \rightarrow \mathbb{R} \) is differentiable for each \( n \in \mathbb{N} \), and that there exist \( c \in (a,b) \) and constants \( M, M' \) such that \( |f_n(c)| \leq M \) and \( |f'_n(x)| \leq M' \) for all \( n \in \mathbb{N} \) and all \( x \in [a,b] \). Prove that \( \{f_n\} \) has a uniformly convergent subsequence.

4. Let \( (f_\alpha)_{\alpha \in A} \) be an equicontinuous family of functions on an interval \([t_1,t_2]\), and suppose that \( \sup_{\alpha \in A} f_\alpha(t) < \infty \) for each \( t \in [t_1,t_2] \). Prove that \( \sup_{\alpha \in A} f_\alpha \) is continuous on \([t_1,t_2]\).

5. Background: It follows from Theorem 2.37 and Problem 2.26 that a subset \( E \) of a metric space \( X \) is compact if and only if it is \(^1\) “countably compact,” where sequential compactness of \( E \) means that every infinite subset of \( E \) has a limit point in \( E \).

Let \( K \) be a compact metric space. Using the above equivalence of compactness and sequential compactness for metric spaces, and using Theorems 7.24–7.25, prove that a subset \( \mathcal{F} \subset C(K) \) (with the metric on \( C(K) \) determined by the supremum norm) is compact if and only if \( \mathcal{F} \) is closed, bounded, and equicontinuous.

Practice problems (not to be handed in):
Chapter 7: #15, 17, 18.

The solutions manual is here: http://digital.library.wisc.edu/1793/67009.

A challenging optional problem: Suppose that \( \{a_n\} \) is a sequence of real numbers with \( a_n \geq a_{n+1} > 0 \) for all \( n \in \mathbb{N} \). Prove that \( \sum_{n=1}^{\infty} a_n \sin(nx) \) converges uniformly on \( \mathbb{R} \) if and only if \( \lim_{n \rightarrow \infty} na_n = 0 \).
(Feel free to post your solution on Piazza. No advice will be given in office hours for this problem; feel free to discuss it with classmates.)

Thanks to Nathan Fugleberg for the following plots:

- the continuous but nowhere differentiable function in Theorem 7.18 (try small \( N \) and zoom in):
  https://www.desmos.com/calculator/ipx1t99lob

- the space-filling curve in Assignment 4 #4 (run the animation):
  https://www.desmos.com/calculator/p2128om85x

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\(^1\)In this problem, this was originally termed “sequentially compact,” whose correct definition is that \( E \) is sequentially compact if every sequence in \( E \) has a convergent subsequence with limit in \( E \). In a metric space, compactness, countable compactness, and sequential compactness are equivalent.