Math 321 Assignment 4: Due Friday, February 2 at start of class

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Midterm 1 will be held in class on Wednesday, February 7. It covers the material on Assignments 1–4; this corresponds to the text pp. 120–134 and pp. 143–150 (to Theorem 7.12).

The obvious Theorem 7.9 was not discussed in class, but it may be useful to appeal to it in this assignment.

You must staple your pages together when you submit your assignments.

1. For \( n = 1, 2, \ldots \) and \( x \in \mathbb{R} \), let \( f_n(x) = \frac{x^{2n}}{1+x^{2n}} \).
   (a) Prove that the limit \( f(x) = \lim_{n \to \infty} f_n(x) \) exists for all \( x \) and determine the limit.
   (b) Prove that the convergence in part (a) is not uniform on \( \mathbb{R} \).
   (c) Determine (with proof) all intervals containing \( x = 0 \) on which the convergence in part (a) is uniform.
   (d) Determine (with proof) all intervals containing \( x = 2 \) on which the convergence in part (a) is uniform.

2. Suppose that \( f_n \) is continuous on \( E \) for all \( n \), \( f_n \) converges uniformly to \( f \) on \( E \), that \( x \) is a limit point of \( E \), and that \( x_n \) is a sequence in \( E \) with \( \lim_{n \to \infty} x_n = x \). Prove that \( \lim_{n \to \infty} f_n(x_n) = f(x) \).

3. For \( n = 1, 2, \ldots, c \in \mathbb{R} \), and \( x \in [0,1] \), let \( f_n(x) = n^c x(1-x^2)^n \).
   (a) Prove that the limit \( f(x) = \lim_{n \to \infty} f_n(x) \) exists for all \( x \in [0,1] \) and determine the limit.
   (b) Determine (with proof) the values of \( c \) for which the convergence in part (a) is uniform.
   (c) Determine (with proof) the values of \( c \) for which \( \int_0^1 f(x) \, dx \) is equal to \( \lim_{n \to \infty} \int_0^1 f_n(x) \, dx \).

4. In this problem, we construct a space-filling curve, i.e., a continuous onto mapping \( f : [0,1] \to [0,1] \times [0,1] \).
   For \( t \in [0,2] \) we define
   \[
   \phi(t) = \begin{cases} 
   0 & (0 \leq t \leq \frac{1}{4}) \\
   3t - 1 & \left(\frac{1}{4} \leq t \leq \frac{3}{4}\right) \\
   1 & \left(\frac{3}{4} \leq t \leq 1\right) \\
   -3t + 5 & \left(\frac{3}{4} \leq t \leq \frac{5}{4}\right) \\
   0 & \left(\frac{5}{4} \leq t \leq 2\right).
   \end{cases}
   \]
   Then we extend \( \phi \) to all of \( \mathbb{R} \) by making it periodic with period 2, i.e., \( \phi(t + 2) = \phi(t) \). The graph of \( \phi \) is a trapezoidal wave (draw the graph). We define \( f_1, f_2 \) on \( \mathbb{R} \) by
   \[
   f_1(t) = \sum_{n=1}^{\infty} \frac{\phi(3^{2n-2}t)}{2^n}, \quad f_2(t) = \sum_{n=1}^{\infty} \frac{\phi(3^{2n-1}t)}{2^n}.
   \]
   (a) Prove that \( f_1 \) and \( f_2 \) are well-defined and continuous on \( \mathbb{R} \) and that \( f_1(t) \in [0,1] \) for all \( t \).
   (b) Let \( (a, b) \in [0,1] \times [0,1] \). We write the binary representation of \( a, b \) as \( a = \sum_{n=1}^{\infty} \frac{a_n}{2^n} \) and \( b = \sum_{n=1}^{\infty} \frac{b_n}{2^n} \) with \( a_n, b_n \in \{0,1\} \). Let \( c = 2 \sum_{n=1}^{\infty} \frac{c_n}{2^n} \) where \( c_{2n-1} = a_n \) and \( c_{2n} = b_n \) for \( n = 1, 2, \ldots \) Prove that \( \phi(3^k c) = c_{k+1} \) for \( k = 0, 1, 2, \ldots \). Hint: consider separately the cases \( c_{k+1} = 0 \) and \( c_{k+1} = 1 \).
   (c) Prove that \( f(c) = (a, b) \).

Practice problems (not to be handed in):
Chapter 7: #1, 2, 3, 4, 5, 6, 7, 8, 9, 10.