Math 321 Assignment 2: Due Friday, January 24 at start of class

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Please read instructions for assignment submission at:

1. Let $f, g$ be bounded functions on $[a, b]$ and let $\alpha$ be an increasing function on $[a, b]$.

(a) Prove that for $c \in (a, b)$,
\[
\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha.
\]

(b) Prove that
\[
\int_a^b (f + g)\,d\alpha \leq \int_a^b f\,d\alpha + \int_a^b g\,d\alpha.
\]

(c) Give an example to show that strict inequality is possible in (b).

2. Prove that there is no bounded function $\rho \in \mathcal{R}$ on $[-1, 1]$ such that $\int_{-1}^1 f(x)\rho(x)\,dx = f(0)$ for every continuous function $f$ on $[-1, 1]$.

**Remark:** In physics and applied mathematics it is common to encounter the Dirac delta function which is purported to be a "function" $\delta$ on $\mathbb{R}$ with the properties that $\delta(x) = 0$ if $x \neq 0$, $\delta(0) = \infty$, and for a continuous function $f$ on $[a, b]$ with $a < 0 < b$ we have $\int_a^b f(x)\delta(x)\,dx = f(0)$. We have not defined integrals of functions that might be infinite.

This problem shows that the $\delta$ function cannot be understood within Riemann integration. Theorem 6.15 shows that the delta function can however be realised by a Riemann–Stieltjes integral with $\alpha$ the unit step function of Definition 6.14. A different mathematically rigorous treatment of the delta function uses the notion of distributions.

3. Consider a wire of length 1 cm (modelled as the interval $[0, 1]$) with constant density $\mu$ gram/cm, containing two point-like beads with masses $m_1$ and $m_2$ grams located at $s_1 = \frac{1}{3}$ and $s_2 = \frac{2}{3}$, respectively. Find a function $\rho : [0, 1] \to \mathbb{R}$ such that the total mass of any portion $(a, b] \subset [0, 1]$ of the wire (with $a < b$) is given by $M(x) = \int_0^x \rho\,dx$.

**Remark:** This shows that the Riemann–Stieltjes integral is well suited to handle a combination of discrete and continuous mass. The notion of "mass" need not be restricted to the one measured in grams. It also applies, e.g., to probability, as in the next problem.

4. This problem shows how the Riemann–Stieltjes integral unifies and extends the concept of expectation for discrete and continuous random variables in probability theory. Let $\Omega$ be a set and let $X : \Omega \to [a, b]$ be a function (a random variable).

[For (b) and (c), look ahead to Rudin 6.20 and 6.21; use these if you wish.]

(a) A discrete random variable is an $X$ such that $X(\Omega)$ is countable (possibly finite), say $X(\Omega) = \{x_i\}$. Suppose that these values $x_i$ all lie in $(a, b)$, and suppose that the probability that $X$ takes the value $x_i$ is $p_i \in [0, 1]$, with $\sum_i p_i = 1$. The expectation of $X$ is defined to be $EX = \sum_i x_i p_i$. Find $F : [a, b] \to [0, 1]$, monotone increasing with $F(a) = 0$ and $F(b) = 1$, such that $EX = \int_a^b x\,dF(x)$.

(b) A continuous random variable with values in $[a, b]$ is an $X$ with the property that the probability that $X \in [a, x]$ is equal to $\int_a^x f(t)\,dt$ for each $x \in [a, b]$, for some $f : [a, b] \to [0, \infty)$ with $f \in \mathcal{R}$ and $\int_a^b f(t)\,dt = 1$. Assume that $f$ is continuous. The expectation of $X$ is defined to be $EX = \int_a^b xf(x)\,dx$. Find $F : [a, b] \to [0, 1]$, monotone increasing with $F(a) = 0$ and $F(b) = 1$, such that $EX = \int_a^b x\,dF(x)$. 
(c) Some random variables are neither discrete nor continuous. For example, consider a random variable $X$ which is equal to 2 with probability $\frac{1}{2}$ and is otherwise uniformly chosen from $[0, 2]$. It is a superposition of a discrete and a continuous random variable. Find $F : [0, 2] \to [0, 1]$ such that it makes sense to say that the expectation of $X$ is $EX = \int_0^2 x dF(x)$. Prove that your integral gives $EX = \frac{3}{2}$.

**Practice problems** (not to be handed in):
Chapter 6: #6, 11, 12.
The solutions manual is here: [http://digital.library.wisc.edu/1793/67009](http://digital.library.wisc.edu/1793/67009).
Prove that
\[
\int_a^b (f + g) d\alpha \geq \int_a^b f d\alpha + \int_a^b g d\alpha.
\]
(This can be seen to follow from #1(b).)