Math 321 Assignment 2: Due Friday, January 19 at start of class

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You must staple your pages together when you submit your assignments.

1. Let \( f \) be a bounded function on \([a, b]\) and let \( \alpha \) be an increasing function on \([a, b]\).
   (a) Prove that for \( c \in (a, b) \),
   \[
   -\int_a^c f \, d\alpha = \int_a^c f \, d\alpha + \int_c^b f \, d\alpha.
   \]
   (b) Prove that
   \[
   \int_a^b (f + g) \, d\alpha \leq \int_a^b f \, d\alpha + \int_a^b g \, d\alpha.
   \]
   (c) Prove that
   \[
   \int_a^b (f + g) \, d\alpha \geq \int_a^b f \, d\alpha + \int_a^b g \, d\alpha.
   \]
   (d) Give an example to show that strict inequality is possible in (b).

2. Prove that there is no bounded function \( \rho \in \mathcal{R} \) on \([-1, 1]\) such that \( \int_{-1}^1 f(x) \rho(x) \, dx = f(0) \) for every continuous function \( f \) on \([-1, 1]\).

   Remark: In physics and applied mathematics it is common to encounter the Dirac delta function which is purported to be a “function” \( \delta \) on \( \mathbb{R} \) with the properties that \( \delta(x) = 0 \) if \( x \neq 0 \), \( \delta(0) = \infty \), and for a continuous function \( f \) on \([a, b]\) with \( a < 0 < b \) we have \( \int_a^b f(x) \delta(x) \, dx = f(0) \). We have not defined integrals of functions that might be infinite. This problem shows that the \( \delta \) function cannot be understood within Riemann integration. Theorem 6.15 shows that the delta function can however be realised by a Riemann-Stieltjes integral with \( \alpha \) the unit step function of Definition 6.14. A different mathematically rigorous treatment of the delta function uses the notion of distributions.

3. This problem shows how the Riemann–Stieltjes integral unifies and extends the concept of expectation for discrete and continuous random variables in probability theory. Let \( \Omega \) be a set and let \( X : \Omega \to [a, b] \) be a function (a random variable). [Hint for (b) and (c): look ahead to Rudin 6.20 and 6.21; use these if you wish.]
   (a) A discrete random variable is an \( X \) such that \( X(\Omega) = \{x_i\} \). Suppose that these values \( x_i \) all lie in \((a, b)\), and suppose that the probability that \( X \) takes the value \( x_i \) is \( p_i \in [0, 1] \), with \( \sum p_i = 1 \). The expectation of \( X \) is defined to be \( EX = \sum x_i p_i \). Find \( F : [a, b] \to [0, 1] \), monotone increasing with \( F(a) = 0 \) and \( F(b) = 1 \), such that \( EX = \int_a^b x \, dF(x) \).
   (b) A continuous random variable is an \( X \) with the property that the probability that \( X \in [a, x] \) is equal to \( \int_a^x f(t) \, dt \) for each \( x \in [a, b] \), for some \( f : [a, b] \to [0, \infty] \) with \( f \in \mathcal{R} \) and \( \int_a^b f(t) \, dt = 1 \). Assume that \( f \) is continuous. The expectation of \( X \) is defined to be \( EX = \int_a^b x f(x) \, dx \). Find \( F : [a, b] \to [0, 1] \), monotone increasing with \( F(a) = 0 \) and \( F(b) = 1 \), such that \( EX = \int_a^b x \, dF(x) \).
   (c) Some random variables are neither discrete nor continuous. For example, consider a random variable \( X \) which is equal to \( 2 \) with probability \( \frac{1}{2} \) and is otherwise uniformly chosen from \([0, 2] \). It is a superposition of a discrete and a continuous random variable. Find \( F : [0, 2] \to [0, 1] \) such that it makes sense to say that the expectation of \( X \) is \( EX = \int_0^2 x \, dF(x) \). Prove that your integral gives \( EX = \frac{3}{2} \).

Practice problems from Rudin (not to be handed in):
Reading assignment: Items 4.28–4.31. Since we are dealing a lot with monotone functions \( \alpha \), these items are useful.
Chapter 6: #6, 7, 8, 11, 12.