Math 321 Assignment 1: Due Friday, January 12 at start of class

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You must staple your pages together when you submit your assignments.

1. Prove directly from the definition of the Riemann–Stieltjes integral that \( \int_a^b f \, d\alpha = \alpha(b) - \alpha(a) \). Here \( \int_a^b f \, d\alpha \) means \( \int_a^b f \, d\alpha \) with \( f(x) = 1 \) for all \( x \in [a, b] \).

2. Let \( s_1, \ldots, s_k \) be distinct points in \( [a, b] \) and set \( S = \{ s_1, \ldots, s_k \} \). Define \( f : [a, b] \to \mathbb{R} \) by \( f(x) = 1 \) if \( x \not\in S \) and \( f(x) = 0 \) if \( x \in S \). Prove that \( f \in \mathcal{R} \) and compute \( \int_a^b f \, dx \).

3. For each of the following cases, the function \( f : [0, 1] \to \mathbb{R} \) obeys \( f(x) = 0 \) for \( x < \frac{1}{2} \) and \( f(x) = 2 \) for \( x > \frac{1}{2} \), and the function \( \alpha : [0, 1] \to \mathbb{R} \) obeys \( \alpha(x) = u \) for \( x < \frac{1}{2} \) and \( \alpha(x) = v \) for \( x > \frac{1}{2} \) with fixed \( u < v \).
   
   (a) Suppose that \( f(\frac{1}{2}) = 0 \) and \( \alpha(\frac{1}{2}) = v \). Prove that \( f \in \mathcal{R}(\alpha) \) and compute \( \int_0^1 f \, d\alpha \).
   
   (b) Suppose that \( f(\frac{1}{2}) = 2 \) and \( \alpha(\frac{1}{2}) = u \). Prove that \( f \in \mathcal{R}(\alpha) \) and compute \( \int_0^1 f \, d\alpha \).
   
   (c) Suppose that \( f(\frac{1}{2}) = 0 \) and \( \alpha(\frac{1}{2}) = u \). Prove that \( f \not\in \mathcal{R}(\alpha) \).

4. Let \( f : [0, 1] \to \mathbb{R} \) be continuous with \( f(x) \geq 0 \) for each \( x \in [0, 1] \). Let \( \alpha(x) = x \).
   
   (a) Suppose that there exists \( x_0 \in [0, 1] \) such that \( f(x_0) > 0 \). Prove that there exists a partition \( P \) and a constant \( c > 0 \) such that \( L(P, f, \alpha) > c \). Conclude that \( \int_0^1 f \, dx > c \).
   
   (b) Suppose instead that \( \int_0^1 f \, dx = 0 \). Conclude that \( f(x) = 0 \) for all \( x \in [0, 1] \).

5. Give an example of a function \( f : [a, b] \to \mathbb{R} \) and an increasing function \( \alpha \) with \( |f| \in \mathcal{R}(\alpha) \) but \( f \not\in \mathcal{R}(\alpha) \).