Math 320 Assignment 9: Due Friday, November 27 at start of class

This is the last assignment to be handed in. Material from the last two weeks of classes is not covered by Assignments 1–9. It will be essential to do some problems on those final topics; suggested problems from Rudin (not to be handed in) will be posted on the course webpage by November 27.

1. Terminology: an onto function is also called a surjection; a 1-1 function is also called an injection; a function which is both a surjection and an injection is called a bijection.

Let $A, B$ be nonempty sets and let $f : A \rightarrow B$ and $g : B \rightarrow A$ be functions. Suppose that the composite function $g \circ f : A \rightarrow A$ is a surjection, and that $f \circ g : B \rightarrow B$ is an injection. Prove that $f, g$ are both bijections.

2. Let $X = \{(x, y) \in \mathbb{R}^2 : (x, y) \neq (0, 0)\}$. Define $f : X \rightarrow \mathbb{R}$ by $f(x, y) = \frac{xy}{x^2 + y^2}$. Prove that $f$ is continuous on $X$ but there is no way to define $f(0, 0)$ to make $f$ continuous on $\mathbb{R}^2$.

3. Consider the subsets $E_1 = (0, 1)$ and $E_2 = [0, \infty)$ of $\mathbb{R}$.

(a) Find a continuous function $f_1 : E_1 \rightarrow \mathbb{R}$ which is not bounded.
(b) Find a continuous function $f_2 : E_2 \rightarrow \mathbb{R}$ which is not bounded.
(c) Find a continuous function $g_1 : E_1 \rightarrow \mathbb{R}$ which is bounded, but which does not attain a maximum value (i.e., there is no $p \in E_1$ with $g_1(x) \leq g_1(p)$ for all $x \in E_1$).
(d) Find a continuous function $g_2 : E_2 \rightarrow \mathbb{R}$ which is bounded, but which does not attain a maximum value.

4. We define the function $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = e^{-x^2}$.

(a) Find an open set $V \subset \mathbb{R}$ such that $f(V)$ is not open.
(b) Find a closed set $F \subset \mathbb{R}$ such that $f(F)$ is not closed.
(c) Find a set $E \subset \mathbb{R}$ such that $f(E)$ is a proper subset of $\overline{f(E)}$.

(The relevance of this comes from Rudin Chapter 4 #2, which ensures that for any continuous function $f : X \rightarrow Y$ and any $E \subset X$, it is the case that $f(E) \subset \overline{f(E)}$.)

5. Let $X, Y$ be metric spaces. Prove that $f : X \rightarrow Y$ is continuous if and only if $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for every subset $B \subset Y$.

6. Let $X$ be a compact metric space, let $Y$ be a metric space, and let $f : X \rightarrow Y$ be continuous. Let $\{F_n\}$ be a sequence of nonempty closed subsets of $X$, which therefore (by Theorem 2.35) are also compact sets. Suppose that $F_n \supset F_{n+1}$ for all $n$. By the Corollary to Theorem 2.36, $\cap_n F_n$ is nonempty.

(a) Prove that $\cap_n f(F_n)$ is nonempty.
(b) Let $y \in \cap_n f(F_n)$. Prove that there is a subsequence $\{F_{n_j}\}$ of $\{F_n\}$ and a corresponding sequence of points $\{x_{n_j}\}$ such that $x_{n_j} \in F_{n_j}$, $f(x_{n_j}) = y$, and $\{x_{n_j}\}$ converges to a point $x$. Prove that $x \in \cap_n F_n$.
(c) Prove that $\cap_n f(F_n) \subset f(\cap_n F_n)$.

Practice problems from Rudin (not to be handed in):
Chapter 4: #1, 2, 3, 4, 5, 6, 7, 18 (an example of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is continuous at every irrational but at no rational!).