Math 320 Assignment 7: Due Friday, November 3 at start of class

1. Let \( \{s_n\} \) be a sequence of real numbers. In class, we defined
\[
\limsup_{n \to \infty} s_n = \inf \{ \sup_{m \geq n} s_m \} = \lim_{n \to \infty} \{ \sup_{m \geq n} s_m \}.
\]
We denote this as \( L = \lim_{n \to \infty} (\sup_{m \geq n} s_m) \). In Definition 3.16, \( s^* = \limsup_{n \to \infty} s_n \) is defined as
the supremum of the set of all subsequential limits of the sequence \( \{s_n\} \).
Prove that \( s^* = L \), assuming that \( L \) is finite.
(You may also like to convince yourself that \( s^* = L \) when \( L = \infty \) or \( L = -\infty \), but do not hand in this part.)

2. Let \( p \geq 0 \). Determine whether \( \sum_n a_n \) converges or diverges for each of the following choices of \( a_n \)
(the answer can depend on the value of \( p \)):
   a. \( a_n = \frac{\sqrt{n+1} - \sqrt{n}}{np} \),
   b. \( a_n = \frac{1}{(\log n)^p} \),
   c. \( a_n = (\frac{n}{n+1})^n \).

3. Suppose that \( a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0 \) and that \( \lim_{n \to \infty} a_n = 0 \). Let \( s_n = \sum_{k=1}^n (-1)^k a_k \).
   a. Prove that if \( n > m \geq 0 \) then \( |s_n - s_m| \leq a_{m+1} \).
   b. Prove that \( \sum_{k=1}^\infty (-1)^k a_k \) converges and that, for all \( n \geq 0 \),
      \[ |\sum_{k=1}^n (-1)^k a_k - s_n| \leq a_{n+1}. \]
      (This shows that for this special sort of alternating series, the error in approximating the infinite
      sum by a partial sum is at most the first omitted term.)

4. Let \( I \) be an uncountable set. For each \( \alpha \in I \), let \( r_\alpha \) be a positive real number. Define
\[ S = \left\{ \sum_{\alpha \in E} r_\alpha : E \subseteq I, \ E \text{ has finite cardinality} \right\}. \]
Prove that \( S \) is not bounded from above.
(This problem shows why we only consider sums of sequences and not sums over uncountable infinite sets.)

Practice problems (not to be handed in):
Chapter 3: #6, 7, 8, 14(d,e), 16, 17, 18, 19, 20.

A. Given a sequence \( \{x_n\} \) of real numbers with \( x_n \geq 1 \) for all \( n \), let \( p_n = \prod_{k=1}^n x_k \). We say that the
infinite product \( \prod_{k=1}^\infty x_k \) converges and equals \( p \) if the sequence \( p_n \) converges to \( p \). Let \( y_n = x_n - 1 \). Prove
that \( \prod_{k=1}^\infty x_k \) converges if and only if \( \sum_{k=1}^\infty y_k \) converges.
(You may use exponential and/or logarithm functions and their basic properties.)

B. In this question, you may use the fact that \( \log(b/a) = \int_a^b \frac{1}{t} \, dt \) and elementary properties of the integral.
   i. Prove that \( \frac{1}{n+1} \leq \log(n+1) - \log n \leq \frac{1}{n} \).
   ii. Let \( a_n = \sum_{k=1}^n \frac{1}{k} - \log n \). Prove that the limit \( \gamma = \lim_{n \to \infty} a_n \) exists.
      (The constant \( \gamma \) is called the Euler-Mascheroni constant. It gives a precise rate of divergence of the
      harmonic series.)

C. This is a problem for those who have done Chapter 3 #24.
   Let \( (X, d) \) be a metric space. Let \( (X^*, \Delta) \) be the completion of \( (X, d) \), as defined in #24(b), and
let \( \varphi : X \to X^* \) be the distance-preserving map defined in #24(d). Prove that if \( (Z, \rho) \) is a complete
metric space and if \( g : X \to Z \) is a distance-preserving map, then there exists a distance-preserving map
\( f : X^* \to Z \) so that \( g = f \circ \varphi \).
In words: \( (X^*, \Delta) \) is the “smallest” complete metric space containing \( X \), in the sense that every
complete metric space \( (Z, \rho) \) containing \( X \) factors through \( (X^*, \Delta) \).