Math 320 Assignment 6: Due Friday, October 25 at start of class

1. In this question, do not use calculus (including derivatives, L'Hôpital's Rule, Taylor expansion). Compute the following limits. Hint: $a - b = \frac{(a^2 - b^2)}{(a + b)}$.
   
   (a) $\lim_{n \to \infty} \left( (n^2 + n)^{1/2} - n \right)$
   
   (b) $\lim_{n \to \infty} \left( (n^2 + 1)^{1/4} - n^{1/2} \right)$
   
   (c) $\lim_{n \to \infty} \left( (n^2 + n)^{1/4} - n^{1/2} \right)$

2. Let $\{s_n\}$ be a sequence in $\mathbb{R}$. For each of the following, prove the statement if it is true or give a counterexample if it is false.
   
   (a) If $\{s_n\}$ converges then $\lim_{n \to \infty} (s_{n+1} - s_n) = 0$.
   
   (b) If $\lim_{n \to \infty} (s_{n+1} - s_n) = 0$ then $\{s_n\}$ converges.

3. Prove that a sequence $\{p_n\}$ in a metric space $X$ converges to a limit $p \in X$ if and only if every subsequence of $\{p_n\}$ converges to $p$.

4. Suppose that $\{s_n\}$ is a sequence in $\mathbb{R}$ with the property that every subsequence has a subsubsequence that converges to $s \in \mathbb{R}$. Prove that $\lim_{n \to \infty} s_n = s$.

5. Show that there is a Cauchy sequence in $\mathbb{Q}$ that does not converge to a limit in $\mathbb{Q}$. (You can use your knowledge about $\mathbb{R}$ in your proof.)

Practice problems from Rudin (not to be handed in):

Chapter 3: #1, 3, 4, 5, 14(a,b,c). Some of these rely on Theorem 3.14 or Definition 3.16.

Problems 24, 25 give an alternate construction of $\mathbb{R}$ from $\mathbb{Q}$, which does not use Dedekind cuts. Instead, it is shown how to form the completion of any metric space $X$ as a new complete metric space $X^*$ whose elements are certain equivalence classes of Cauchy sequences in $X$. The real numbers $\mathbb{R}$ can be regarded as $\mathbb{Q}^*$.

The solutions manual is here: [http://digital.library.wisc.edu/1793/67009](http://digital.library.wisc.edu/1793/67009).

Here are two more practice problems (solutions not provided):

1. (a) Find a sequence $\{r_n\}$ of real numbers such that the set of its subsequential limits is $[0, 1]$. (Be sure to prove that your sequence has the desired property.)

   (b) Prove that there is no sequence $\{r_n\}$ of real numbers such that the set of its subsequential limits is $(0, 1)$.

2. Prove that a Cauchy sequence $\{p_n\}$ in a metric space $X$ must be bounded.