Math 320 Assignment 3: Due Friday, September 29 at start of class

1. Recall the definition of a cut, from the Appendix to Chapter 1. For \( n \in \mathbb{N} \), we define cuts \( \alpha_n \) by \( \alpha_n = \{ p \in \mathbb{Q} : p < 1 - \frac{1}{n} \} \). Let \( A = \{ \alpha_n : n \in \mathbb{N} \} \). Determine (with proof) the least upper bound and greatest lower bound of \( A \).

2. Read Definitions 1.3, 2.1, 2.2, 2.9. Suppose \( A, B \) are sets and \( f \) is a function from \( A \) to \( B \) as in Definition 2.1.
   (a) Prove that, for all subsets \( B_1, B_2 \) of \( B \):
   i. \( f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2) \).
   ii. \( f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2) \).
   (b) Let \( A_1, A_2 \) be subsets of \( A \). Prove or give a counterexample for the following statements.
      i. \( f(A_1 \cup A_2) = f(A_1) \cup f(A_2) \).
      ii. \( f(A_1 \cap A_2) = f(A_1) \cap f(A_2) \).

3. Let \( z, w \in \mathbb{C} \) with \( \bar{z}w \neq 1 \). Prove that:
   (a) \[ \left| \frac{z - w}{1 - \bar{z}w} \right| < 1 \] if \( |z| < 1 \) and \( |w| < 1 \),
   (b) \[ \left| \frac{z - w}{1 - \bar{z}w} \right| = 1 \] if \( |z| = 1 \) or \( |w| = 1 \).

   (You may find it useful to first prove that there is an \( r \geq 0 \) and \( v \in \mathbb{C} \) with \( |v| = 1 \) such that \( z = rv \), and use this to reduce to the case of \( z \) real and positive. For the first case, elementary calculus can be used to show that \( (r - w)(r - \bar{w}) < (1 - rw)(1 - r\bar{w}) \).)

4. Read Definition 2.15. Suppose that \( X \) is a metric space with metric \( d \). For each of the following, prove that the proposal defines a metric on \( X \) or give a counterexample to prove that it does not.
   (a) \( \rho(p, q) = (d(p, q))^2 \).
   (b) \( m(p, q) = \min\{d(p, q), 1\} \).

Practice problems (not to be handed in):

Rudin, Chapter 1: \#1, 2, 3, 5, 8, 9, 10, 13, 15, 16, 17, 18. Chapter 2: \#11.

1. (a) Given \( q \in \mathbb{Q} \), prove that \( \{ p \in \mathbb{Q} : p < q \} \) is a cut.
   (b) Prove that \( \{ p \in \mathbb{Q} : p^2 < 2 \} \) is not a cut.
   (c) Prove that \( \{ p \in \mathbb{Q} : p^2 < 2 \} \cup \{ p \in \mathbb{Q} : p \leq 0 \} \) is a cut.

2. Let \( X \) be a metric space with metric \( d \), and let \( X' \) be another metric space with metric \( d' \). Show that the following define metrics on the Cartesian product \( X \times X' \):
   (a) \( d_1((p, p'), (q, q')) = d(p, q) + d'(p', q') \),
   (b) \( d_2((p, p'), (q, q')) = \max\{d(p, q), d'(p', q')\} \),
   (c) \( d_3((p, p'), (q, q')) = ([d(p, q)]^2 + [d'(p', q')]^2)^{1/2} \).