Math 320 Assignment 2: Due Friday, September 20 at start of class

1. Let $S$ be the set of all finite length strings of letters from the alphabet $a$–$z$. Let $<$ be the lexicographic order on $S$. This means that if $w = a_1a_2...a_k$ and $v = b_1b_2...b_\ell$ are two strings, then $w < v$ if either of the following two things hold (A): $k < \ell$ and $a_i = b_i$ for each $i = 1,\ldots,k$, or (B) The letter $a_j$ comes before $b_j$ in the alphabet, where $j$ is the smallest index where $a_i \neq b_i$. Thus for example, $a < aa$, $aa < b$, and $b < cde$.

(a) Let $E \subset S$ be the set of all finite strings that begin with the character $a$. What is the least upper bound for $E$? Prove that your answer is correct.

(b) Does $S$ have the least upper bound property? If so, prove it. If not, find an example showing that $S$ does not have the least upper bound property and prove that your example is correct.

2. Recall that elements of $\mathbb{R}$ are cuts, where a cut is a set $\alpha \subset \mathbb{Q}$ such that:

1. $\alpha \neq \emptyset, \alpha \neq \mathbb{Q}$
2. $p \in \alpha \implies q \in \alpha$ for all $q < p$
3. $p \in \alpha \implies \exists r \in \alpha$ such that $p < r$

We define “$\alpha < \beta$” to mean that $\alpha \subset \beta$, $\alpha \neq \beta$, and we define $\alpha + \beta = \{r+s : r \in \alpha, s \in \beta\}$. Prove directly from these definitions that if $\alpha$, $\beta$, $\gamma$, $\delta \in \mathbb{R}$, then

$$\alpha < \beta \text{ and } \gamma < \delta \implies \alpha + \gamma < \beta + \delta.$$  

3. Let $\mathbb{Q}$ and $\mathbb{R}$ be the ordered set of rational numbers and real numbers, respectively, with their usual ordering. Prove that $f(x) = x$ is the only function $f : \mathbb{Q} \to \mathbb{R}$ satisfying the following properties:

- $f$ is an injection.
- For all $a, b \in \mathbb{Q}$, $f(a+b) = f(a) + f(b)$ and $f(a \cdot b) = f(a) \cdot f(b)$.

Hint: think about $f(1)$.

4. Fix a real number $b > 1$ and a natural number $n$. Recall that, by definition, $b^{\frac{1}{n}}$ is the unique $x > 0$ that obeys $x^n = b$. Prove that, by making $n$ sufficiently large, $b^{\frac{1}{n}}$ can be made arbitrarily close to 1 by completing the following outline.

(a) Prove that $b^n - 1 \geq n(b - 1)$.

(b) Prove that $b - 1 \geq n(b^{\frac{1}{n}} - 1)$.

(c) Let $t \in \mathbb{R}$ satisfy $t > 1$, and choose $n \in \mathbb{N}$ such that $n > \frac{b-1}{t}$. Prove that $b^{\frac{1}{n}} < t$.

Recommended problems: The following problems from Rudin are recommended for practice:
Chapter 1, #6,7,8. Do not hand in your solutions. The solutions manual is here: http://digital.library.wisc.edu/1793/67009.