Math 320 Assignment 1: Due Friday, September 13 at start of class

Please read instructions for assignment submission at:

1. Let \( p \) be a prime number. Prove that there is no rational number whose square is \( p \). (You may appeal to the Fundamental Theorem of Arithmetic.)

2. Let \( E = \{ r \in \mathbb{Q} : r^3 > 8 \} \). Find inf \( E \) and sup \( E \) if they exist, and determine whether they are elements of \( E \). (Prove all your claims.)

3. Let \((S, <)\) be an ordered set and \( A \subset S \). Suppose that \( A \) contains a largest element, which we denote by max \( A \).
   (a) Prove that \( A \) has exactly one largest element, i.e., that max \( A \) is unique.
   (b) Prove that sup \( A \) exists in \( S \) and equals max \( A \).

4. Read the section on Fields, pp.5–8. In this problem we study a set that satisfies the field axioms but does not satisfy the order axioms. Consider the field \( \mathbb{F}_3 \). This field has three elements, which we will call 0, 1, 2. (Do not confuse these elements with real numbers: 0, 1 are the elements prescribed to exist by axioms (A4) and (M4), and 2 is an arbitrary name for a third element.) Addition and multiplication are defined by the following addition and multiplication tables:

\[
\begin{array}{c|ccc}
+ & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 1 & 2 & 0 \\
2 & 2 & 0 & 1 \\
\end{array}
\quad
\begin{array}{c|ccc}
\times & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 \\
2 & 0 & 2 & 1 \\
\end{array}
\]

Using a proof by contradiction, show that it is impossible to define an operation “\(<\)” that satisfies the order axioms. Hint: Proposition 1.18(d).

Remark. \( \mathbb{F}_3 \) is an example of a finite field. Finite fields play an important role in algebra, number theory, and computer science.

5. The real numbers \( \mathbb{R} \) are constructed in Theorem 1.19 as an ordered field which has the least-upper-bound property.

Find the sup and inf of each of the following sets of real numbers:
   (a) All numbers of the form \( 2^{-p} + 3^{-q} + 5^{-r} \), where \( p, q, r \) each take on all positive integer values.
   (b) \( E = \{ x : 3x^2 - 10x + 3 < 0 \} \).
   (c) \( E = \{ x : (x-a)(x-b)(x-c)(x-d) < 0 \} \), where \( a < b < c < d \).

6. Let \( S_1 \) and \( S_2 \) be nonempty subsets of \( \mathbb{R} \) that are bounded above. Let \( S_1 + S_2 = \{ x+y : x \in S_1, y \in S_2 \} \) and \( S_1 - S_2 = \{ x-y : x \in s_1, y \in S_2 \} \). For each of the following statements, give a proof if it is true or a counterexample if it is false.
   (a) sup(\( S_1 + S_2 \)) = sup \( S_1 \) + sup \( S_2 \).
   (b) If \( S_2 \) is also bounded below then sup(\( S_1 - S_2 \)) = sup \( S_1 \) - sup \( S_2 \).

Recommended problems: The following problems from Rudin are recommended for practice: Chapter 1, #1,2,3,4,5. Do not hand in your solutions. The solutions manual is here: http://digital.library.wisc.edu/1793/67009.