I. Problems to be handed in:

1. A particle of mass 1 g has a random velocity $X$ that is uniformly distributed between 2 cm/s and 3 cm/s.
   
   (a) Find the cumulative distribution function of the particle’s kinetic energy $T = \frac{1}{2} X^2$.
   
   (b) Find the probability density function of $T$.
   
   (c) Determine the mean and standard deviation of $T$ in two ways:
       
       (i) using the p.d.f. of $T$ from part (b),
       
       (ii) using the Law of the Unconscious Statistician and the uniform density of $X$.

2. Let $X_1, X_2, \ldots, X_n$ be independent random variables, each with uniform distribution on $(0, 1)$. Let $M$ be the minimum of these random variables.
   
   (a) Show that the cumulative distribution function of $M$ is $F_M(x) = 1 - (1 - x)^n$, $0 \leq x \leq 1$.
   
   (b) Find the probability density function of $M$.
   
   (c) Determine the mean and variance of $M$.

3. Let $X$ be a uniform random variable on $[0, 1]$ and let $Y$ be uniform on $[-1, 0]$. Compute Cov($X, X^2$) and Cov($Y, Y^2$).
   
   (The first is positive and the second is negative, consistent with the fact that $X^2$ increases when $X$ increases and $X^2$ decreases when $X$ decreases, whereas the opposite is true for $Y$ and $Y^2$.)

4. The time $T$ (in hours past noon) until the arrival of the first taxi has Exponential(5) distribution, and the time $B$ until the first bus is independent with Exponential(3) distribution.
   
   (a) Write down the joint probability density function of $T$ and $B$.
   
   (b) Find the probability that the first taxi arrives before the first bus.
   
   (c) If you arrive at noon and take the first bus or taxi (whichever arrives first), what is the distribution of your waiting time (give the name and parameter(s))? [Hint: let $X = \min(T, B)$, and find $P(X > y)$.]

5. This question considers uniform random points on the disc $0 \leq x^2 + y^2 \leq 1$ with radius 1 centred at the origin. Parts (d) and (e) require Octave; the command unifrnd may be helpful — this function generates continuous uniform random numbers on a specified interval. As usual, type help unifrnd for details.
   
   (a) A point is uniformly chosen in the unit disk $0 \leq x^2 + y^2 \leq 1$. Find the probability that its distance from the origin is less than $r$, for $0 \leq r \leq 1$.
   
   (b) Compute its expected distance from the origin.
   
   (c) Let the co-ordinates of the point be $(X, Y)$. Determine the marginal p.d.f. of $X$. Are $X$ and $Y$ independent?
   
   (d) One way to generate uniform random points on this disc is to first generate uniform random points on the square with corners $(\pm 1, \pm 1)$, by independently generating their $x$ and $y$ coordinates as uniform random variables on $(-1, 1)$, and then ignore the points outside the unit circle. To visualize this, generate 5000 uniform random points on the square with corners $(\pm 1, \pm 1)$ and create a scatterplot of their locations. By hand (or in Octave, if you know how to do
Chapter 2: 41

\[ \sigma^2 \leq \frac{1}{2} \] 

\[ \sigma(X) \sigma(P) \geq \frac{h}{2} \] 

II. Recommended problems: These provide additional practice but are not to be handed in.

A. Chapter 2: 41[2(n − 1)p(1 − p)], 43[n/(m + 1)], 50, 51[r/p], 57.

B. Chapter 5: 1 \[ b \in [e^{-1}, e^{-1}]; \] 2 \[ 6/\mu]; 3, 4 \[ assume that all service times are independent; 0, \frac{1}{2\pi}, \frac{1}{2} \] 

C. Optional problem for those interested in quantum mechanics and the uncertainty principle:

Consider a quantum mechanical system in state \( \psi \), where \( \psi \) is vector in the Hilbert space consisting of complex-valued square-integrable functions, with inner product \( \langle \psi_1, \psi_2 \rangle = \int \psi_1(x)^* \psi_2(x) dx \). Assume that \( \psi \) is normalized, so that \( \langle \psi, \psi \rangle = \int |\psi(x)|^2 dx = 1 \). Observables such as position \( X \) or momentum \( P \) are represented by self-adjoint linear operators on the Hilbert space. The expected value of an observable \( A \) is given by \( \langle \psi, A\psi \rangle = \int \psi(x)^* (A\psi)(x) dx \). The standard deviation \( \sigma(A) \) of a measurement of the observable \( A \) is given by \( \sigma(A)^2 = \langle \psi, (A - \langle \psi, A\psi \rangle)^2 \psi \rangle \). It is a general mathematical theorem that for any self-adjoint linear operators \( A \) and \( B \), with commutator \( [A, B] = AB - BA \),

\[ \langle \psi, A^2 \psi \rangle \langle \psi, B^2 \psi \rangle \geq \frac{1}{4} |\langle \psi, [A, B] \psi \rangle|^2. \]

(You can prove this by adapting the proof of the Cauchy–Schwarz inequality, or see Lemma IV.6.1 of E. Prugovečki, Quantum Mechanics in Hilbert Space, Academic Press, 1971.)

The commutator of the position and momentum operators is \( [X, P] = i\hbar \). Using the above, show that \( \sigma(X)\sigma(P) \geq \hbar/2 \). This is a statement of the uncertainty principle.
Quote of the week: When she was rid of the pretense of paper and pen, phrase-making and biography, she turned her attention in a more legitimate direction, though, strangely enough, she would rather have confessed her wildest dreams of hurricane and prairie than the fact that, upstairs, alone in her room, she rose early in the morning or sat up late at night to . . . work at mathematics. No force on earth would have made her confess that. Her actions when thus engaged were furtive and secretive, like those of some nocturnal animal . . . Perhaps the unwomanly nature of the science made her instinctively wish to conceal her love of it. But the more profound reason was that in her mind mathematics were directly opposed to literature. She would not have cared to confess how infinitely she preferred the exactitude, the star-like impersonality, of figures to the confusion, agitation, and vagueness of the finest prose.

Virginia Woolf, in Night and Day