Math 303 Assignment 7: Due Friday, March 10 at start of class

I. Problems to be handed in:

1. Let \{N(t) : t \geq 0\} be a Poisson process of rate \(\lambda\), and let \(S_n\) denote the time of the \(n\)th event. Find:
   
   (a) \(E(N(5))\)
   
   (b) \(E(S_3)\)
   
   (c) \(P(N(5) < 3)\)
   
   (d) \(P(S_3 > 5)\)
   
   (e) \(P(S_3 > 5 | N(2) = 1)\).

2. Arrivals of the Number 14 bus form a Poisson process of rate 4 buses per hour and arrivals of the Number 99 bus form an independent Poisson process of rate 10 buses per hour.
   
   (a) What is the probability that exactly 12 buses arrive in one hour?
   
   (b) What is the probability that exactly 3 Number 99 buses arrive while I wait for a Number 14 bus?
   
   (c) The maintenance depot goes on strike and the number of all buses is reduced by 50%. In this case, what is the probability that I wait 15 minutes without seeing a single bus?

3. Determine, with complete proof, which of the following functions are \(o(h)\) as \(h \to 0\), and which are not:
   
   (a) \(\sqrt{h}\), (b) \(h^{3/2}\), (c) \(e^{h^2} - 1\), (d) \(he^h\), (e) \(\sin h - h\).

4. Smith has a small booth where he sells lottery tickets. Customers arrive according to a Poisson process of rate \(\lambda = 1\) per minute. He will close the shop on the first occasion that \(a = 6\) minutes have elapsed since the last customer arrived. Show that, on average, he keeps the shop open for approximately 6 hours and 42 minutes. Indeed, show that the general formula for the average time is \(\lambda^{-1}(e^{\lambda a} - 1)\).
   
   Hint: condition on the arrival time of the first customer, i.e., use (3.2b) ((3.3b) in 10th ed.).
   
   For some review of computing expectation by conditioning, it may be useful to consult Example 3.12 (3.13 in 10th ed.).

5. Smith wishes to cross a single lane of fast-moving traffic. Vehicles pass by according to a Poisson process of rate \(\lambda\), and suppose it takes time \(a\) to walk across the lane. Suppose also that Smith can correctly foresee the times at which vehicles will pass by.
   
   (a) Determine the average time it takes to cross the lane safely.
       
       Hint: Apply the previous problem.
   
   (b) How long on average would it take to cross two similar lanes when he must walk straight across (he will not cross if, at any time while crossing, a car would pass in either direction).
   
   (c) How long on average would it take to cross two similar lanes when an island in the middle of the road makes it safe to stop half-way?
   
   (d) Given \(\lambda > 0\), is there any value of \(a > 0\) for which it is better not to have an island?

6. In the lottery shop of #4, let \(X\) denote the first time when the gap between the arrival of two consecutive customers is less than \(a\). Find \(EX\), as a function of \(\lambda\) and \(a\), and for the special cases \(\lambda = 1, a = 6\) and \(\lambda = 1, a = \frac{1}{6}\).
II. Recommended problems: These provide additional practice but are not to be handed in.
Chapter 5 #36*, 37 $[\mu^{-1} + \lambda^{-1}]$, 51 $\int_0^t (1 + \alpha(t - s)) \beta e^{-\beta s} ds$, 53 $[e^{-1}, e^{-1} + \frac{1}{5} e^{-2}]$, 57*.

Quote of the week: “I think you’re begging the question,” said Haydock, “and I can see looming ahead one of those terrible exercises in probability where six men have white hats and six men have black hats and you have to work it out by mathematics how likely it is that the hats will get mixed up and in what proportion. If you start thinking about things like that, you would go round the bend. Let me assure you of that!”

Agatha Christie in The Mirror Crack’d