Math 303 Assignment 4: Due Friday, February 3 at start of class

Reminder: Test 1 will be held in class on Wednesday February 8, and will be based on the material covered in Assignments 1–4. No assignment will be given on February 3; Assignment 5 will be available on February 10.

I. Problems to be handed in:

1. Five balls are distributed between urn $A$ and urn $B$. At each step, an urn is chosen at random and a ball from that urn is moved to the other urn, if the chosen urn is not empty. If the chosen urn is empty, no ball is moved. The state of the system is the number of balls in urn $A$.

   (a) Determine the transition matrix of this Markov Chain.
   (b) Determine its stationary distribution.
   (c) If urn $A$ is now empty, how long must we wait on average until it is empty for the next time?

2. Determine the stationary distribution of the Markov Chain which is given by the urn model of Assignment 1, #5. (The “guess and verify” method is recommended.)

3. Five white balls and five black balls are distributed in two urns in such a way that each urn contains five balls. We say that the system is in state $i$ ($i = 0, 1, 2, 3, 4, 5$) if the first urn contains $i$ white balls. At each step, we draw one ball from each urn, place the ball drawn from the first urn in the second urn, and place the ball drawn from the second urn in the first urn. Let $X_n$ denote the state of the system after the $n$th step. This defines a Markov chain.

   (a) Calculate the one-step transition matrix for this Markov chain.
   (b) Determine the stationary distribution for the chain. (The “guess and verify” method is recommended.)
   (c) Suppose the first urn contains five white balls. How long, on average, will it take until the next time the first urn contains five white balls?

4. You have 3 books $B_1, B_2, B_3$ that you keep on a shelf. You frequently choose one of them to read. You choose $B_1$ with probability $\frac{1}{2}$, and choose each of $B_2, B_3$ with probability $\frac{1}{4}$. After you have finished with a book, you return it to the left position on the shelf. Regard this as a Markov chain whose state space consists of the 3! possible orderings of your books on the shelf.

   (a) Determine the transition matrix for this Markov Chain.
   (b) Determine its stationary distribution. (The “guess and verify” method is recommended.)

5. Let $X_n$ be a Markov Chain with state space $\{0, 1, 2, 3, \ldots\}$, with transition probabilities $P_{n,n-1} = 1$ for $n \geq 1$, and $P_{0,n} = c_a(n+1)^{-a}$ for $n \geq 0$, with $a > 1$, and $c_a$ chosen so that $\sum_{n=0}^{\infty} c_a(n+1)^{-a} = 1$.

   (a) Let $T$ be the time until the first return to 0, starting from 0. Determine the p.m.f. of $T$. For which values of $a$ is the expectation $ET$ finite?
   (b) For which values of $a$ is the chain recurrent? For which values is it positive recurrent? For which values is it null recurrent?
   (c) In the positive recurrent case, determine the stationary distribution.

II. Recommended problems: These provide additional practice but are not to be handed in.

Chapter 4 #42, 52 [37], 54, 68b*, 71, 73.

Quote of the week: I graduated from Douglass College without distinction. I was in the top 98% of my class and damn glad to be there. I slept in the library and daydreamed my way through history lecture. I failed math twice, never fully grasping probability theory. I mean, first off, who cares if you pick a black ball or a white ball out of the bag? And second, if you’re bent over about the color, don’t leave it to chance. Look in the damn bag and pick the color you want.

Stephanie Plum in Hard Eight by Janet Evanovich