Tutorial 5. The Simon-Lieb Inequality and the Infinite Volume Limit.

Let $\Gamma = (V, E)$ be a locally finite graph. We are primarily interested in the two cases of the hypercubic lattice $\Gamma = \mathbb{Z}^d$, and the discrete torus $\Gamma = \mathbb{Z}^d/L\mathbb{Z}^d$ of side length $L \in \mathbb{N}$. Let E_x be the expectation of a simple random walk $(X_n)_{n\geq 0}$ on Γ starting at $x \in V$, and $I_{m,n}$ be the number of self-intersections of X between times m and n,

$$I_{m,n} = \sum_{m\leq i<j\leq n} 1_{X_i = X_j}, \quad I_n = I_{0,n}. \tag{1}$$

The two-point function of the weakly self-avoiding in the domain $D \subset V$ is defined by

$$G_{\kappa,D}(x,y) = \sum_{n\geq 0} E_x(e^{-gI_n} 1_{X_n=y, n<T_D}) e^{-\kappa n}, \quad x, y \in V,\ \kappa \in \mathbb{R}, \tag{2}$$

where $T_D = \inf\{n \geq 0 : X_n \notin U\}$ is the exit time of D. We define the closure $\bar{D} = D \cup \partial D = \{x \in V : x \in D \text{ or there exists } y \in D \text{ s.t. } x \sim y\}$. The two-point function on the whole graph is written as G_κ, without reference to D. Let $c_n(x, y) = E_x(e^{-gI_n} 1_{X_n=y})$, $c_n = \sum_{y\in V} c_n(0, y)$, and define the susceptibility by

$$\chi(\kappa) = \sum_{y\in V} G_\kappa(0, y) = \sum_{n\geq 0} c_n e^{-\kappa n}. \tag{3}$$

Problem 1. Verify that $(c_n)_{n\geq 0}$ is a submultiplicative sequence, i.e. $c_{n+m} \leq c_n c_m$, and conclude that $\frac{1}{n}\log(c_n)$ converges to its infimum, which is e^{κ_c} by definition. In particular, notice that for $\kappa < \kappa_c$, $\chi(\kappa) = \infty$ and for $\kappa > \kappa_c$, $\chi(\kappa) < \infty$.

Problem 2. Let $\chi^L(\kappa)$ be the susceptibility for $\mathbb{Z}^d/L\mathbb{Z}^d$, and $\chi(\kappa)$ be the susceptibility for $\mathbb{Z}^d$. Prove that $\chi^L(\kappa) \leq \chi(\kappa)$ for $L \geq 3$, and, in particular, that $\kappa_c(\mathbb{Z}^d) \geq \kappa_c(\mathbb{Z}^d/L\mathbb{Z}^d)$. Here, $\kappa_c(\Gamma)$ denotes the critical point of the weakly self-avoiding walk on the graph Γ.

Problem 3. Prove the following version of the Simon-Lieb inequality[1] for the discrete-time weakly self-avoiding walk on a graph $\Gamma = (V, E)$. Let $D \subset V$ be a subset and show that

$$G_\kappa(x, y) - G_{\kappa,D}(x, y) \leq \sum_{z\in\partial D} G_{\kappa,\bar{D}}(x, z) G_\kappa(z, y). \tag{4}$$

Note that if $x \in D$ and $y \in D^c$, then $G_{\kappa,D}(x, y) = 0$.

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[1]B. Simon [1] and E. H. Lieb [2] have proved an analogous inequality for the ferromagnetic Ising model on $\mathbb{Z}^d$. If $(\sigma_x)_{x\in\mathbb{Z}^d}$ is the spin field of the Ising model, $\langle\,\cdot\,\rangle$ its expectation, $\langle\,\cdot\,\rangle_D$ the expectation of the spin system inside $D \subset \mathbb{Z}^d$, $x, y \in \mathbb{Z}^d$, $A \subset \mathbb{Z}^d$ is a set that separates x and y, and $A^x \subset \mathbb{Z}^d$ is the connected component of $\mathbb{Z}^d \setminus A$ that contains x (but not y), then

$$\langle\sigma_x\sigma_y\rangle \leq \sum_{z\in A^x} \langle\sigma_x\sigma_z\rangle_{A^x} \cdot \langle\sigma_z\sigma_y\rangle.$$

Also, for the continuous-time weakly self-avoiding walk, a similar inequality holds. Its statement is slightly different, but the essential idea is the same. We restrict to the case of discrete time to keep it simple.

Problem 4. Let $\Lambda_L = \{-L, \ldots, L\}^d \subset \mathbb{Z}^d$ with $\partial\Lambda_L = \{x \in \mathbb{Z}^d : |x|_\infty = L+1\}$. Then, for $\kappa > \kappa_c$, $\sum_{y\in\mathbb{Z}^d} G_\kappa(0,y)$ is finite, and thus $\theta := \sum_{y\in\partial\Lambda_L} G_\kappa(0,y) < 1$ for L sufficiently large. Conclude from Problem 3 with $D = \Lambda_L$ that for $|y|_\infty > L+1$,

$$G_\kappa(0,y) \le \theta^{\lfloor |y|_\infty/(L+1)\rfloor} \sup_{x\in\mathbb{Z}^d} G_\kappa(0,x). \tag{5}$$

Problem 5. Let $(T_L)_{L\in\mathbb{N}}$ be a sequence of discrete tori, $T_L = (V_L, E_L)$, with the vertex sets embedded in $\mathbb{Z}^d$ by $V_L = \Lambda_L$ where Λ_L is as in Problem 4; in particular, $V_L \subset V_{L+1}$. Let G_κ^L be the two-point function on T_L, and G_κ be the two-point function on $\mathbb{Z}^d$. Use Problem 2 and Problem 4 to prove that for all $\kappa > \kappa_c \equiv \kappa_c(\mathbb{Z}^d)$, $x, y \in \mathbb{Z}^d$,

$$G_\kappa^L(x,y) \to G_\kappa(x,y) \quad \text{as } L \to \infty. \tag{6}$$

Conclude that

$$G_{\kappa_c}(x,y) = \lim_{\kappa\downarrow\kappa_c} \lim_{L\to\infty} G_\kappa^L(x,y). \tag{7}$$

References

[1] B. Simon. Correlation inequalities and the decay of correlations in ferromagnets. *Comm. Math. Phys.*, **77**:111–126, (1980).

[2] E.H. Lieb. A refinement of Simon's correlation inequality. *Comm. Math. Phys.*, **77**:127–136, (1980).