

Critical correlation functions for the 4-dimensional weakly self-avoiding walk and n -component $|\varphi|^4$ model

Gordon Slade* and Alexandre Tomberg*

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Abstract

We extend and apply a rigorous renormalisation group method to study critical correlation functions, on the 4-dimensional lattice \mathbb{Z}^4 , for the weakly coupled n -component $|\varphi|^4$ spin model for all $n \geq 1$, and for the continuous-time weakly self-avoiding walk. For the $|\varphi|^4$ model, we prove that the critical two-point function has $|x|^{-2}$ (Gaussian) decay asymptotically, for $n \geq 1$. We also determine the asymptotic decay of the critical correlations of the squares of components of φ , including the logarithmic corrections to Gaussian scaling, for $n \geq 1$. The above extends previously known results for $n = 1$ to all $n \geq 1$, and also observes new phenomena for $n > 1$, all with a new method of proof. For the continuous-time weakly self-avoiding walk, we determine the decay of the critical generating function for the “watermelon” network consisting of p weakly mutually- and self-avoiding walks, for all $p \geq 1$, including the logarithmic corrections. This extends a previously known result for $p = 1$, for which there is no logarithmic correction, to a much more general setting. In addition, for both models, we study the approach to the critical point and prove existence of logarithmic corrections to scaling for certain correlation functions. Our method gives a rigorous analysis of the weakly self-avoiding walk as the $n = 0$ case of the $|\varphi|^4$ model, and provides a unified treatment of both models, and of all the above results.

1 Introduction and main results

1.1 Introduction

The subject of critical phenomena and phase transitions in statistical physics has been an important source of problems and inspiration for mathematics for over half a century, especially in probability theory and combinatorics. Fundamental models of statistical mechanics, such as the Ising model, the $O(n)$ model, the $|\varphi|^4$ model, the self-avoiding walk, percolation, the random cluster model, dimers, and others, have become increasingly prominent in mathematics and now form the raw material for large and diverse research communities.

*Department of Mathematics, University of British Columbia, Vancouver, BC, Canada V6T 1Z2.
slade@math.ubc.ca, atomberg@math.ubc.ca

For ferromagnetic spin systems such as the Ising and $|\varphi|^4$ models, the phase transition results from an interplay between a local ferromagnetic interaction which causes spins to tend to align, and thermal fluctuations which tend to destroy long-range order. As the temperature decreases past its critical value, long-range order emerges suddenly and in a singular manner. Universal critical exponents and scaling limits are associated with the phase transition. Part of the fascination of the subject is due to the variation of behaviour as the underlying spatial dimension d changes.

The case of $d = 2$ is particularly beautiful and rich, and is connected with conformal invariance. Magnificent advances in our rigorous understanding of phase transitions for $d = 2$ have emerged following the invention of the Schramm–Loewner Evolution. Nevertheless, significant challenges remain. For example, while we now have good understanding of the critical behaviour of site percolation on the triangular lattice [68–70], no corresponding results have been obtained for bond percolation on the square lattice. Although there has been recent progress, e.g., [31,47], important issues concerning universality remain to be resolved.

The physically most interesting case of $d = 3$ is the most difficult and the least understood. Very recently, and about 70 years after Onsager’s exact solution of the 2-dimensional Ising model, it was proved that the spontaneous magnetisation of the 3-dimensional Ising model vanishes at the critical temperature [6]. However, the vanishing of the percolation probability for $d = 3$ remains one of the most significant open questions in probability theory, and generally the calculation of critical exponents for $d = 3$ is an enormous challenge. An interesting exception is the exact solution for 3-dimensional branched polymers in [22,53].

In high dimensions, much is known. For the Ising and 1- and 2-component $|\varphi|^4$ models, methods involving reflection positivity have led to proofs of mean-field behaviour for nearest-neighbour interactions in dimensions $d > 4$ [4,42]. Such methods have been extended to show that in the upper critical dimension $d = 4$, deviations from mean-field behaviour are at worst logarithmic for some quantities [7–9,16], although these references do not prove that logarithmic corrections do exist as predicted in the physics literature. Lace expansion methods have been used to prove mean-field behaviour in dimensions greater than 4, for the self-avoiding walk [49], for spread-out Ising models [65], for weakly coupled or spread-out 1-component φ^4 [64], among other models [67].

Much of the attention has been devoted to models with discrete symmetry such as the Ising model, where the group \mathbb{Z}_2 acts on the interaction by flipping all spins simultaneously. However, from a physical perspective, continuous symmetry is highly relevant, and from a mathematical perspective, it describes richer phenomena. The most basic non-trivial examples of such models are the $O(n)$ and $|\varphi|^4$ models, which generalise the Ising model, and in which the interaction between n -component spins is invariant under the orthogonal group. It has long been understood that the n -component $|\varphi|^4$ model can be obtained as a limit of $O(n)$ models [33,66], and that the converse holds is an elementary fact. The existence of a phase transition in nearest-neighbour models, in which the continuous $O(n)$ symmetry is spontaneously broken, has been proved in all dimensions $d > 2$ using the *infrared bound*. Although elegant, this method has limitations: it is limited to reflection-positive models and does not supply detailed understanding of the behaviour at the critical point.

Our subject in this paper is the critical behaviour of the continuous-time weakly self-avoiding walk (or WSAW), and of the n -component $|\varphi|^4$ model, for all $n \geq 1$, in the upper critical dimension $d = 4$. Over forty years ago, de Gennes observed that n -component spin models formally correspond to the self-avoiding walk in the limit $n \rightarrow 0$ [45]. Since the number of components is a natural number, the limit $n \rightarrow 0$ is mathematically undefined, at least naively. However, using

the basis developed in [11, 12, 15], we are able to interpret WSAW in a mathematically rigorous manner as the $n = 0$ case of the n -component $|\varphi|^4$ model, and provide a unified treatment for all $n \geq 0$. In particular, our results also apply to the case $n \geq 2$ of continuous symmetry.

The basis we build upon is a rigorous renormalisation group method. In physics, the renormalisation group has been used simultaneously to explain the existence of universality and to compute the universal quantities associated with critical phenomena. A nonrigorous analysis of the $|\varphi|^4$ model is worked out in [73], and the model is sometimes referred to as the *Landau–Ginzburg–Wilson* model. A rigorous renormalisation group method is applied to the $|\varphi|^4$ model in [15]. This method applies to WSAW once the model is rewritten in terms of a supersymmetric integral representation [25, 60, 62]. The supersymmetric representation we use is in terms of a 2-component boson field and a 2-component fermion field; the former contributes a factor 2 to each loop, and the latter -2 , with the net effect that loops do not contribute. In [12, 15], the renormalisation group method is applied to prove that for $n \geq 0$ the susceptibility diverges at the critical point as $\varepsilon^{-1}(\log \varepsilon^{-1})^{(n+2)/(n+8)}$, with $n = 0$ corresponding to WSAW and $n \geq 1$ corresponding to $|\varphi|^4$. This confirms predictions of [17, 54, 72]. For $n = 1$, it was proved much earlier in [48, 50]. Also, in [11], $|x|^{-2}$ decay is proved for the critical two-point function of the 4-dimensional WSAW.

This last result required the introduction of *observables* to the analysis, and one of our major themes is to extend the variety of observables considered, and to apply the formalism of observables also to the $|\varphi|^4$ model. The latter was not done in [15]. (Somewhat related ideas were used in [32]; different methods were developed in [38].) Moreover, we develop new techniques concerning reduction of the $O(n)$ symmetry of the $|\varphi|^4$ model, for $n \geq 2$.

A lesson learned from [12, 15] is that if we set $n = 0$ in the second-order perturbative calculations used for the rigorous renormalisation group analysis of the $|\varphi|^4$ model, then what results is exactly the second-order perturbative calculations in the rigorous renormalisation group analysis for WSAW. This is a rigorous version of the observation of de Gennes [45]. A general theory developed in [28, 29] permits a unified treatment of non-perturbative effects. Consequently, our main task here is to carry out appropriate perturbative calculations, with an appeal to the general theory to bound all the error terms. We do this in a unified way for all $n \geq 0$, *including* $n = 0$.

With this approach, we derive the asymptotic decay of several critical correlation functions, in dimension $d = 4$. For $|\varphi|^4$, we prove $|x|^{-2}$ decay for the critical two-point function for all $n \geq 1$. This extends previous results for $n = 1$ due to [40, 43, 44], to all $n \geq 1$. In [44], it was also shown that for $n = 1$ the critical correlation between φ_0^2 and φ_x^2 decays as $|x|^{-4}(\log |x|)^{-2/3}$. We extend this to general $n \geq 1$. In addition, we prove that for the multi-component case of $n \geq 2$, at the critical point there is positive correlation between same field components at distant points, but negative correlation between different field components. Related results are obtained for logarithmic corrections to scaling for correlations of fields, as the critical point is approached. For WSAW, we obtain the decay of the critical “watermelon” networks, consisting for fixed $p \geq 1$ of p weakly mutually- and self-avoiding walks joining two distant sites, at the critical point. This extends the result for $p = 1$ obtained in [11]. (An earlier related result for $p = 1$ is [52], for a model which is neither a lattice model nor a model containing walks, but which nevertheless shares features in common with WSAW.) For $p \geq 2$, we also determine the logarithmic corrections to scaling for “star networks” consisting of p weakly mutually- and self-avoiding walks which intersect at the origin, as the critical point is approached.

We next give precise definitions of the $|\varphi|^4$ and WSAW models, followed by precise statements of our results.

1.2 The $|\varphi|^4$ model

1.2.1 Definition of the model

Let $L > 1$ be an integer, and let $\Lambda = \Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$ be the d -dimensional discrete torus of side length L^N . Ultimately we are interested in the thermodynamic limit $N \rightarrow \infty$. For convenience, we sometimes consider Λ to be a box in \mathbb{Z}^d , approximately centred at the origin, without opposite sides identified to create the torus. We can then regard fixed $a, b \in \mathbb{Z}^d$ as points in Λ provided that N is large enough, and we make this identification throughout the paper. In particular, we always assume that N is sufficiently large that Λ contains given a, b .

The spin field φ is a function $\varphi : \Lambda \rightarrow \mathbb{R}^n$, or equivalently a vector $\varphi \in (\mathbb{R}^n)^\Lambda$. We use subscripts to index $x \in \Lambda$ and superscripts for the component $i = 1, \dots, n$. We write $|v|$ for the Euclidean norm $|v|^2 = \sum_{i=1}^n (v^i)^2$ and $v \cdot w = \sum_{i=1}^n v^i w^i$ for the Euclidean inner product on \mathbb{R}^n . For $e \in \mathbb{Z}^d$ with $|e| = 1$, we define the discrete gradient by $(\nabla^e \varphi)_x = \varphi_{x+e} - \varphi_x$, and the discrete Laplacian by $\Delta = -\frac{1}{2} \sum_{e \in \mathbb{Z}^d: |e|=1} \nabla^{-e} \nabla^e$. We write $\varphi_x \cdot (-\Delta \varphi)_x = \sum_{i=1}^n \varphi_x^i (-\Delta \varphi^i)_x$.

Given $g > 0, \nu \in \mathbb{R}$, we define a function $U_{g,\nu,N}$ of the field by

$$U_{g,\nu,N}(\varphi) = \sum_{x \in \Lambda} \left(\frac{1}{4} g |\varphi_x|^4 + \frac{1}{2} \nu |\varphi_x|^2 + \frac{1}{2} \varphi_x \cdot (-\Delta \varphi_x) \right). \quad (1.1)$$

By definition, the quartic term is $|\varphi_x|^4 = (\varphi_x \cdot \varphi_x)^2$. Then we define the expectation of a random variable $F : (\mathbb{R}^n)^\Lambda \rightarrow \mathbb{R}$ by

$$\langle F \rangle_{g,\nu,N} = \frac{1}{\mathcal{Z}_{g,\nu,N}} \int F(\varphi) e^{-U_{g,\nu,N}(\varphi)} d\varphi, \quad (1.2)$$

where $d\varphi$ is the Lebesgue measure on $(\mathbb{R}^n)^\Lambda$, and $\mathcal{Z}_{g,\nu,N}$ is a normalisation constant (the *partition function*) defined so that $\langle 1 \rangle_{g,\nu,N} = 1$. Thus φ is a field of classical continuous n -component spins on the torus Λ , i.e., with periodic boundary conditions.

The *susceptibility* is defined as the limit

$$\chi(g, \nu; n) = \lim_{N \rightarrow \infty} \sum_{x \in \Lambda_N} \langle \varphi_a^1 \varphi_x^1 \rangle_{g,\nu,N} = n^{-1} \lim_{N \rightarrow \infty} \sum_{x \in \Lambda_N} \langle \varphi_a \cdot \varphi_x \rangle_{g,\nu,N}. \quad (1.3)$$

By translation-invariance of the measure, χ is independent of $a \in \mathbb{Z}^d$. For $n = 1, 2$, standard correlation inequalities [41] imply that for the case of free boundary conditions the limit defining the susceptibility exists (possibly infinite) and is monotone non-increasing in ν . Proofs are lacking for $n > 2$ due to a lack of correlation inequalities in this case (as is discussed, e.g., in [41]), although one expects that these facts known for $n \leq 2$ should remain true also for $n > 2$. In our theorems below, we prove the existence of the infinite volume limit with periodic boundary conditions directly in the situations covered by the theorems, without application of any correlation inequalities.

For $d = 4$, for small $g > 0$, and for all $n \geq 1$, it is proved in [15] that there is a critical value $\nu_c(g, n)$ such that, for $\nu = \nu_c + \varepsilon$, the susceptibility diverges according to the asymptotic formula

$$\chi(g, \nu; n) \sim A_{g,n} \varepsilon^{-1} (\log \varepsilon^{-1})^{(n+2)/(n+8)} \quad \text{as } \varepsilon \downarrow 0, \quad (1.4)$$

for some amplitude $A_{g,n} > 0$. Here, and throughout the paper, we write $f \sim g$ to mean $\lim f/g = 1$. In this paper, we study correlation functions both *exactly at* the critical value $\nu_c(g, n)$ and in the

limit as $\nu \downarrow \nu_c(g, n)$. It is also shown in [15] that $\nu_c(g, n) = -\mathbf{a}g + O(g^2)$ with $\mathbf{a} = (n + 2)G_{00} > 0$, where, for $a, b \in \mathbb{Z}^4$, G_{ab} denotes the massless lattice Green function. From an analytic perspective,

$$G_{ab} = (-\Delta_{\mathbb{Z}^4}^{-1})_{ab}, \quad (1.5)$$

where the right-hand side is the matrix element of the inverse lattice Laplacian acting on scalar functions on \mathbb{Z}^4 . From a probabilistic perspective, G_{ab} equals $\frac{1}{2d}$ times the expected number of visits to b of simple random walk on \mathbb{Z}^4 started from a (the extra factor $\frac{1}{2d} = \frac{1}{8}$ is due to our definition of the Laplacian). It is a standard fact (see, e.g., [56]) that, as $|a - b| \rightarrow \infty$,

$$G_{ab} = \frac{1}{(2\pi)^2|a - b|^2} \left(1 + O\left(\frac{1}{|a - b|^2}\right) \right). \quad (1.6)$$

1.2.2 Correlation functions

We study correlation functions in the infinite volume limit. The existence of the infinite volume limit is not known for general n , and it is part of our results that the limit does exist for $n \geq 1$, provided g is sufficiently small. We write $\langle F \rangle_{g, \nu} = \lim_{N \rightarrow \infty} \langle F \rangle_{g, \nu, N}$ when the limit exists. We also write $\langle F; G \rangle = \langle FG \rangle - \langle F \rangle \langle G \rangle$, both in finite and infinite volume, for the *correlation* or *truncated expectation* of F, G . Our main results include the precise asymptotic behaviour as $|a - b| \rightarrow \infty$, for all $n \geq 1$ and for $p = 1, 2$, of the 4-dimensional infinite volume *critical* truncated correlation functions

$$\langle (\varphi_a^i)^p; (\varphi_b^j)^p \rangle_{g, \nu_c} \quad \text{for } 1 \leq i, j \leq n. \quad (1.7)$$

By the $O(n)$ symmetry, it is sufficient to consider the two special cases $(i, j) = (1, 1)$ and $(i, j) = (1, 2)$. The first case turns out to be positive. Since the transformation $\varphi^1 \mapsto -\varphi^1$ does not change the measure, the second case is zero for all odd p . The second case only makes sense for $n \geq 2$, and it turns out to be *negative* for $p = 2$. In principle our methods could be used to study also $p > 2$, but new issues arise for $p > 2$ and we have not pursued this case.

We define the critical correlation functions (1.7) as the limit

$$\langle (\varphi_a^i)^p; (\varphi_b^j)^p \rangle_{g, \nu_c} = \lim_{\varepsilon \downarrow 0} \lim_{N \rightarrow \infty} \langle (\varphi_a^i)^p; (\varphi_b^j)^p \rangle_{g, \nu_c + \varepsilon, N}. \quad (1.8)$$

Similarly, for $\nu > \nu_c$, we write

$$\sum_{x_1, x_2 \in \mathbb{Z}^4} \langle \varphi_{x_1}^i \varphi_{x_2}^j; (\varphi_a^k)^2 \rangle_{g, \nu} = \lim_{N \rightarrow \infty} \sum_{x_1, x_2 \in \Lambda_N} \langle \varphi_{x_1}^i \varphi_{x_2}^j; (\varphi_a^k)^2 \rangle_{g, \nu, N}. \quad (1.9)$$

It is part of the statement of our results that these limits exist for small $g > 0$ and for $n \geq 1$, $p = 1, 2$. However, we do require that the limit be taken through tori $\Lambda_N = \mathbb{Z}^4 / L^N \mathbb{Z}^4$ with L large, as this restriction is part of the hypotheses of results from [12, 15, 28, 29] upon which our analysis relies. We therefore always tacitly assume that L is large, throughout the rest of the paper, for both the $|\varphi|^4$ and WSAW models. When we assume that g is small in theorems, g is chosen small depending on the value of L , and depending also on $n \geq 0$.

1.3 The WSAW model

1.3.1 Definition of the model

Let X be the continuous-time simple random walk on the integer lattice \mathbb{Z}^d , with $d > 0$. In more detail, X is the stochastic process with right-continuous sample paths that takes its steps at the times of the events of a rate- $2d$ Poisson process. Steps are taken uniformly at random to one of the $2d$ nearest neighbours of the current position, and are independent both of the Poisson process and of all other steps. Let E_a denote the expectation for the process with $X(0) = a \in \mathbb{Z}^d$. The *local time of X at x up to time T* is the random variable $L_T(x) = \int_0^T \mathbb{1}_{X(t)=x} dt$, and the *intersection local time up to time T* is the random variable

$$I(T) = \int_0^T \int_0^T \mathbb{1}_{X(t_1)=X(t_2)} dt_1 dt_2 = \sum_{x \in \mathbb{Z}^d} (L_T(x))^2. \quad (1.10)$$

Given $g > 0$, $\nu \in \mathbb{R}$, and $a, b \in \mathbb{Z}^d$, the continuous-time weakly self-avoiding walk *two-point function* is defined by the integral (possibly infinite)

$$W_{ab}^{(1)}(g, \nu) = \int_0^\infty E_a \left(e^{-gI(T)} \mathbb{1}_{X(T)=b} \right) e^{-\nu T} dT. \quad (1.11)$$

In (1.11), self-intersections are suppressed by the factor $e^{-gI(T)}$. The connection between (1.11) and the two-point function of the usual strictly self-avoiding walk is discussed in [19]. In dimension 4, (1.11) is also known as the two-point function of the lattice *Edwards model* (with continuous time).

We set $c_{g,T} = E_a(e^{-gI(T)})$, and define the *susceptibility* by

$$\chi(g, \nu; 0) = \sum_{b \in \mathbb{Z}^d} W_{ab}^{(1)}(g, \nu) = \int_0^\infty c_{g,T} e^{-\nu T} dT. \quad (1.12)$$

By translation-invariance of the simple random walk and of (1.10), $c_{g,T}$ and χ are independent of the point $a \in \mathbb{Z}^d$. A standard subadditivity argument [12] shows that for all dimensions $d > 0$ there exists a *critical value* $\nu_c = \nu_c(g; 0) \in (-\infty, 0]$ (depending also on d) such that

$$\chi(g, \nu; 0) < \infty \quad \text{if and only if} \quad \nu > \nu_c. \quad (1.13)$$

It is shown in [12] that for $d = 4$, for small $g > 0$ and for $\nu = \nu_c + \varepsilon$, the susceptibility diverges as

$$\chi(g, \nu; 0) \sim A_{g,0} \varepsilon^{-1} (\log \varepsilon^{-1})^{1/4} \quad \text{as } \varepsilon \downarrow 0. \quad (1.14)$$

Moreover, $\nu_c(g; 0) = -ag + O(g^2)$ with $\mathbf{a} = 2G_{00} > 0$.

The above asymptotic formulas for the susceptibility and critical point are both consistent with setting $n = 0$ in the corresponding statements for the $|\varphi|^4$ model in Section 1.2.1.

1.3.2 Watermelon and star networks

For $p \geq 1$, consider the vector of p independent continuous-time simple random walks on \mathbb{Z}^4 :

$$X(T) = (X^1(T_1), \dots, X^p(T_p)) \quad \text{for } T = (T_1, \dots, T_p) \in \mathbb{R}_+^p. \quad (1.15)$$

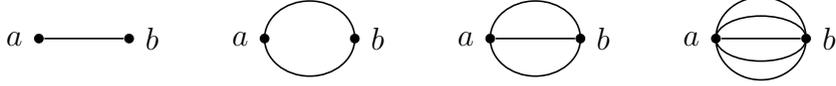


Figure 1: Watermelon networks for $p = 1, 2, 3, 5$.

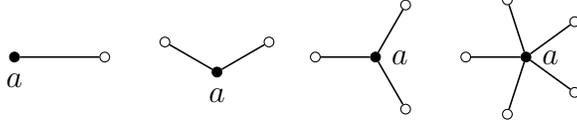


Figure 2: Star networks for $p = 1, 2, 3, 5$.

We write E_a for the expectation of X with $X_k(0) = a$ for all k . We define the corresponding local times, for $x \in \mathbb{Z}^4$, by

$$L_{T_k}^k(x) = \int_0^{T_k} \mathbb{1}_{X_k(t)=x} dt \quad \text{and} \quad L_T(x) = L_{T_1}^1(x) + \cdots + L_{T_p}^p(x). \quad (1.16)$$

Let $I_p(T) = \sum_{x \in \mathbb{Z}^4} (L_T(x))^2$. We write $X(T) = b$ to mean that $X^k(T_k) = b$ for all $k = 1, \dots, p$, and write $dT = dT_1 \cdots dT_p$. The p -watermelon network is then defined by

$$W_{ab}^{(p)}(g, \nu) = p! \int_{\mathbb{R}_+^p} E_a [e^{-gI_p(T)} \mathbb{1}_{X(T)=b}] e^{-\nu \|T\|_1} dT. \quad (1.17)$$

The 1-watermelon network is simply the two-point function, which was studied in [11]. There it was proved that the critical two-point function obeys $W_{ab}^{(1)}(g, \nu_c) \sim C|a - b|^{-2}$ for small g . This is as the same asymptotic behaviour (1.6) for the Green function. By definition, $I_p(T) \geq \sum_{i=1}^p I_1^i(T_i)$, where the superscript i indicates the self-intersection local time of X^i . This implies that $W_{ab}^{(p)}(g, \nu_c) \leq p!(W_{ab}^{(1)}(g, \nu_c))^p \leq O(|a - b|^{-2p})$. In particular, the critical p -watermelon is finite for all $p \geq 1$. Our main results provide precise asymptotics for $W_{ab}^{(p)}(g, \nu_c)$ for all $p \geq 1$.

For $p \geq 1$ and $a \in \mathbb{Z}^4$, we also define

$$S^{(p)}(g, \nu) = p! \sum_{b_1, \dots, b_p \in \mathbb{Z}^d} \int_{\mathbb{R}_+^p} E_a [e^{-gI_p(T)} \mathbb{1}_{X^1(T)=b_1} \cdots \mathbb{1}_{X^p(T)=b_p}] e^{-\nu \|T\|_1} dT. \quad (1.18)$$

The right-hand side is independent of a by translation invariance. By definition, $S^{(1)}$ is the susceptibility χ , while, for $p \geq 2$, $S^{(p)}$ is the generating function for a *star network* of weakly self- and mutually-avoiding walks as depicted in Figure 2. By a similar argument to the one employed above for watermelon networks, $S^{(p)}(g, \nu) < p!\chi^p(g, \nu)$. In particular, $S^{(p)}(g, \nu) < \infty$ for $\nu > \nu_c$.

1.4 Main results

Let $n \geq 0$ and $p \geq 1$ be integers. We fix $g > 0$ small and drop it from the notation. Exponents on logarithms turn out to be expressed in terms of

$$\gamma_{n,p}^+ = \binom{p}{2} \frac{n+2}{n+8}, \quad \gamma_{n,p}^- = \binom{p}{2} \frac{2}{n+8}, \quad (1.19)$$

with $\binom{1}{2} = 0$ so that in the degenerate case $\gamma_{n,1}^+ = \gamma_{n,1}^- = 0$. By definition, for $n = 0$ we have $\gamma_{0,p}^+ = \gamma_{0,p}^- = \frac{1}{4}\binom{p}{2}$.

Theorem 1.1. *Let $d = 4$. Let $n \geq 0$ and $p \geq 2$ be integers. Let $g > 0$ be sufficiently small, depending on n, p , and let $\varepsilon = \nu - \nu_c(g; n) > 0$. There are g -dependent constants $A_{n,p,\pm} > 0$ such that the following hold as $\varepsilon \downarrow 0$.*

(i) For $p \geq 2$,

$$\frac{1}{\chi(\nu; 0)^p} S^{(p)}(\nu) \sim \frac{A_{0,p,+}}{(\log \varepsilon^{-1})^{\gamma_{0,p}^+}}. \quad (1.20)$$

(ii) For $p = 2$,

$$\frac{1}{\chi^2} \sum_{x_1, x_2 \in \mathbb{Z}^4} \langle (\varphi_{x_1} \cdot \varphi_{x_2}); |\varphi_a|^2 \rangle_\nu \sim \frac{n A_{n,2,+}}{(\log \varepsilon^{-1})^{\gamma_{n,2}^+}} \quad (n \geq 1), \quad (1.21)$$

$$\frac{1}{\chi^2} \sum_{x_1, x_2 \in \mathbb{Z}^4} \langle \varphi_{x_1}^1 \varphi_{x_2}^1; (\varphi_a^1)^2 \rangle_\nu \sim \frac{n-1}{n} \frac{A_{n,2,-}}{(\log \varepsilon^{-1})^{\gamma_{n,2}^-}} \quad (n \geq 2), \quad (1.22)$$

$$\frac{1}{\chi^2} \sum_{x_1, x_2 \in \mathbb{Z}^4} \langle \varphi_{x_1}^1 \varphi_{x_2}^1; (\varphi_a^2)^2 \rangle_\nu \sim -\frac{1}{n} \frac{A_{n,2,-}}{(\log \varepsilon^{-1})^{\gamma_{n,2}^-}} \quad (n \geq 2). \quad (1.23)$$

(iii) The amplitudes obey, as $g \downarrow 0$,

$$A_{n,p,\pm} = p! (\mathbf{b}g)^{-\gamma_{n,p}^\pm} (1 + O(g)). \quad (1.24)$$

For the case $n \geq 1$, it is part of the statement of the following theorem that the critical correlation functions on \mathbb{Z}^4 exist in the sense of (1.8). We write error estimates as $|a - b| \rightarrow \infty$ in terms of

$$\mathcal{E}_{ab}^{(p)} = \begin{cases} O((\log |a - b|)^{-1}) & (p = 1) \\ O((\log \log |a - b|)(\log |a - b|)^{-1}) & (p \geq 2). \end{cases} \quad (1.25)$$

We also define the constant

$$\mathbf{b} = \frac{n+8}{16\pi^2}. \quad (1.26)$$

Theorem 1.2. *Let $d = 4$. Let $n \geq 0$ and $p \geq 1$ be integers. Let $g > 0$ be sufficiently small, depending on n, p . There are g -dependent constants $A'_{n,p,\pm} > 0$ such that the following hold as $|a - b| \rightarrow \infty$.*

(i) For $p \geq 1$,

$$W_{ab}^{(p)}(\nu_c(0)) = \frac{A'_{0,p,+}}{(\log |a - b|)^{2\gamma_{0,p}^+}} \frac{1}{|a - b|^{2p}} \left(1 + \mathcal{E}_{ab}^{(p)}\right). \quad (1.27)$$

(ii) For $n \geq 1$ and $p = 1, 2$,

$$\langle \varphi_a^1; \varphi_b^1 \rangle_{\nu_c(n)} = \frac{A'_{n,1,+}}{|a - b|^2} \left(1 + \mathcal{E}_{ab}^{(1)}\right), \quad (1.28)$$

$$\langle |\varphi_a|^2; |\varphi_b|^2 \rangle_{\nu_c(n)} = \frac{n A'_{n,2,+}}{(\log |a - b|)^{2\gamma_{n,2}^+}} \frac{1}{|a - b|^4} \left(1 + \mathcal{E}_{ab}^{(2)}\right). \quad (1.29)$$

(iii) For $n \geq 2$ and $p = 2$,

$$\langle (\varphi_a^1)^2; (\varphi_b^1)^2 \rangle_{\nu_c(n)} = \frac{1}{n} \left(\frac{(n-1)A'_{n,2,-}}{(\log|a-b|)^{2\gamma_{n,2}^-}} + \frac{A'_{n,2,+}}{(\log|a-b|)^{2\gamma_{n,2}^+}} \right) \frac{1}{|a-b|^4} \left(1 + \mathcal{E}_{ab}^{(2)} \right), \quad (1.30)$$

$$\langle (\varphi_a^1)^2; (\varphi_b^2)^2 \rangle_{\nu_c(n)} = \frac{1}{n} \left(-\frac{A'_{n,2,-}}{(\log|a-b|)^{2\gamma_{n,2}^-}} + \frac{A'_{n,2,+}}{(\log|a-b|)^{2\gamma_{n,2}^+}} \right) \frac{1}{|a-b|^4} \left(1 + \mathcal{E}_{ab}^{(2)} \right). \quad (1.31)$$

(iv) The amplitudes obey, as $g \downarrow 0$,

$$A'_{n,p,\pm} = \frac{p!}{(2\pi)^{2p}} (\mathbf{b}g)^{-2\gamma_{n,p}^\pm} (1 + O(g)). \quad (1.32)$$

In Theorem 1.2, the interesting asymptotic behaviour as $|a-b| \rightarrow \infty$ is stressed. However, our proof applies more generally, and gives the following result for the case $a = b$, which provides a natural continuity statement as $g \downarrow 0$.

Theorem 1.3. *Let $d = 4$. Let $n \geq 0$ and $p \geq 1$ be integers. Let $g > 0$ be sufficiently small, depending on n, p . Then, as $g \downarrow 0$,*

$$W_{aa}^{(p)}(\nu_c(0)) = G_{aa}^p (p! + O(g)) \quad (p \geq 1), \quad (1.33)$$

$$\langle \varphi_a^1; \varphi_a^1 \rangle_{\nu_c(n)} = G_{aa} (1 + O(g)) \quad (n \geq 1), \quad (1.34)$$

$$\langle |\varphi_a|^2; |\varphi_a|^2 \rangle_{\nu_c(n)} = G_{aa}^2 (2!n + O(g)) \quad (n \geq 1), \quad (1.35)$$

$$\langle (\varphi_a^1)^2; (\varphi_a^2)^2 \rangle_{\nu_c(n)} = O(g) \quad (n \geq 2). \quad (1.36)$$

It is worth mentioning that even to prove that the left-hand sides of (1.27)–(1.31) or (1.33)–(1.36) are *finite* is a nontrivial result. For example, it remains an open problem to prove that the generating function for self-avoiding polygons is finite at the critical point in dimensions $d = 2, 3$ (see [59, Section 8.1]); this is analogous to $W_{aa}^{(1)}(\nu_c(0))$.

For $p = 1$, the right-hand sides of (1.27)–(1.28) give simply $A'_{n,1,+}|a-b|^{-2}(1 + O(\log|a-b|)^{-1})$. The decay of this particular correlation function, namely the *critical two-point function*, is usually written in terms of the critical exponent η as $|a-b|^{-(d-2+\eta)}$, so this is a statement that η takes its mean-field value $\eta = 0$ for all $n \geq 0$, with no logarithmic correction to the leading behaviour. The power $|a-b|^{-2}$ arises in our analysis via the right-hand side of (1.6).

Special cases of Theorem 1.2 have been proven previously. For $(n, p) = (0, 1)$, (1.27) is the main result of [11]; a related result for a model that involves neither a lattice nor walks appears in [52]. For $(n, p) = (1, 1)$, (1.28) is the main result of [43]. For $(n, p) = (1, 2)$, (1.29) was proved in [44]; in this case the leading behaviour is $|a-b|^{-4}(\log|a-b|)^{-2/3}$. For a related model in which an ultraviolet cutoff replaces the lattice setting, a version of (1.28) for the case $(n, p) = (1, 1)$ appears in [40]. The results: (i) (1.27) for $p \geq 2$, (ii) (1.28)–(1.29) for $n \geq 2$ and $p = 1, 2$, and (iii) (1.30)–(1.31) for $n \geq 2$ and $p = 2$, are new as rigorous results.

Concerning (1.32), the factor $(2\pi)^{-2p}$ arises from the p^{th} power of the Green function via (1.6). The power of g in (1.32) matches the power of the logarithm in the term where the amplitude appears. The combination $g \log|a-b|$ is natural since there are no logarithmic corrections for the Gaussian case $g = 0$.

The exponents $\gamma_{n,p}^\pm$ in Theorem 1.1 and the exponents $2p, 2\gamma_{n,p}^\pm$ in Theorem 1.2 are predicted to be universal. In particular, the $n = 1$ exponents of (1.21) and (1.28)–(1.29) are predicted to apply to the Ising model, and the exponents of (1.21)–(1.23) and (1.28)–(1.31) for $n \geq 2$ are predicted to apply to the $O(n)$ model, including the classical XY (or rotor) model for $n = 2$, and the classical Heisenberg model for $n = 3$.

Similarly, the $n = 0, p \geq 1$ case of (1.20) and (1.27), namely (with $\binom{1}{2} = 0$),

$$\frac{1}{\chi(\nu)^p} S^{(p)}(\nu) \sim \frac{A_{0,p,+}}{(\log \varepsilon^{-1})^{\frac{1}{4}\binom{p}{2}}}, \quad W_{ab}^{(p)}(\nu_c) \sim \frac{A'_{0,p,+}}{|a-b|^{2p}(\log |a-b|)^{\frac{2}{4}\binom{p}{2}}}, \quad (1.37)$$

are predicted to apply to the 4-dimensional strictly self-avoiding walk. For $p \geq 2$ *independent* WSAWs, $\chi^{-p} S^{(p)}$ is identically equal to 1, and $W_{ab}^{(p)}(\nu_c)$ is asymptotic to a multiple of $|a-b|^{-2p}$. The logarithmic corrections in (1.37) for p weakly mutually-avoiding walks are consistent with the interpretation that the intersection of each of the $\binom{p}{2}$ pairs of walks at a vertex gives rise to a penalty $(\log \varepsilon^{-1})^{-1/4}$ or $(\log |a-b|)^{-\frac{1}{4}}$ paid by each pair for joining, despite their penchant to avoid. Related results were obtained via a non-rigorous renormalisation analysis in [34], and a detailed non-rigorous general treatment of polymer networks, including also dimensions below 4, can be found in [36]. For the case of simple random walk, the formula for star networks in (1.37) is reminiscent of the fact, proved in [57], that p independent simple random walks started from the origin in \mathbb{Z}^4 do not have pairwise intersections before leaving the ball of radius n , with probability asymptotic to $(\log n)^{-\frac{1}{2}\binom{p}{2}}$ (see [35] for a non-rigorous renormalisation analysis). A number of authors have studied related matters for the case of two simple random walks [5, 39, 55, 63]. For spread-out models of strictly SAW in dimensions $d > 4$, rigorous results for arbitrary graphical networks were obtained in [51]. These results for $d > 4$ include a statement analogous to (1.37) for all $p \geq 1$, but there is no logarithmic correction and the asymptotic behaviour is simply $\text{const}|a-b|^{-p(d-2)}$. See also [59, Theorem 1.5.5] for nearest-neighbour strictly SAW for $d \geq 6$.

For the case $n \geq 2$ and $p = 2$, since $\gamma_{n,2}^- = \frac{2}{n+8} < \frac{n+2}{n+8} = \gamma_{n,2}^+$, Theorem 1.2 gives (for $i \neq j$)

$$\langle (\varphi_a^i)^2; (\varphi_b^i)^2 \rangle_{\nu_c(n)} \sim \frac{n-1}{n} \frac{A'_{n,2,-}}{|a-b|^4 (\log |a-b|)^{4/(n+8)}}, \quad (1.38)$$

$$\langle (\varphi_a^i)^2; (\varphi_b^j)^2 \rangle_{\nu_c(n)} \sim -\frac{1}{n} \frac{A'_{n,2,-}}{|a-b|^4 (\log |a-b|)^{4/(n+8)}}. \quad (1.39)$$

On the other hand, by (1.29),

$$\langle |\varphi_a|^2; |\varphi_b|^2 \rangle_{\nu_c(n)} \sim n \frac{A'_{n,2,+}}{|a-b|^4 (\log |a-b|)^{2(n+2)/(n+8)}}. \quad (1.40)$$

Thus, for an individual component, $(\varphi_a^i)^2$ is more highly correlated with $(\varphi_b^i)^2$, than is $|\varphi_a|^2$ with $|\varphi_b|^2$, due to cancellations with the negative correlations of $(\varphi_a^i)^2$ with $(\varphi_b^j)^2$ for $i \neq j$. Negative correlations for different components at the *same* point are to be expected since $\langle |\varphi_a|^2 \rangle_{\nu_c(n)} < \infty$ by (1.34), and therefore the field has a typical size, so making one component large must come at the cost of making one component small. This is similar to the fact that the squares of different components of a uniform random variable on the sphere are negatively correlated by the length constraint. Our results show how this effect persists over long distances at the critical point.

In the physics literature, $|\varphi|^2$ is referred to as the *energy operator*, so (1.40) gives the asymptotic behaviour of the energy operator correlation. We are not aware of any reference from the physics literature where (1.38)–(1.40) are stated, though it has been observed that the reduction of symmetry (as in the left-hand sides of (1.38)–(1.39) compared to (1.40)) can lead to a change in critical exponents [2, 30]. Reduction of symmetry plays an important role in our proof of (1.38)–(1.40): the $O(n)$ invariant case (1.40) has a different renormalisation group flow than the non-invariant cases (1.38)–(1.39).

In [15], the asymptotic behaviour of the *specific heat*

$$c_H(\nu) = \frac{1}{4} \sum_{b \in \mathbb{Z}^4} \langle |\varphi_a|^2; |\varphi_b|^2 \rangle_{\nu_c + \varepsilon} \quad (1.41)$$

(with the infinite volume limit defined similarly to (1.9)) is studied in the limit $\varepsilon \downarrow 0$. It is proved in [15], confirming predictions of [54, 72], that for $d = 4$, for small $g > 0$, and for $n \geq 1$, there exists $D(n) = D(g, n) > 0$ such that, as $\varepsilon \downarrow 0$,

$$c_H(\nu_c + \varepsilon) \sim D(n) \begin{cases} (\log \varepsilon^{-1})^{(4-n)/(n+8)} & (n = 1, 2, 3) \\ \log \log \varepsilon^{-1} & (n = 4) \\ 1 & (n > 4). \end{cases} \quad (1.42)$$

Interestingly, it was pointed out in [54], where (1.42) was first derived non-rigorously, that the universal aspects of the phase transition for the 4-dimensional $|\varphi|^4$ model with $n = 1$ should also apply to the phase transition in a 3-dimensional uniaxial ferroelectric substance, and the $\frac{4-1}{1+8} = \frac{1}{3}$ power in (1.42) was subsequently confirmed experimentally for the dipolar Ising ferromagnet LiTbF_4 in [3]. The result (1.42) is complemented by (1.40), which implies that, as $R \rightarrow \infty$,

$$\sum_{b \in \mathbb{Z}^4: |b| \leq R} \langle |\varphi_a|^2; |\varphi_b|^2 \rangle_{\nu_c(n)} \sim c(n) \begin{cases} (\log R)^{(4-n)/(n+8)} & (n = 1, 2, 3) \\ \log \log R & (n = 4) \\ 1 & (n > 4). \end{cases} \quad (1.43)$$

Neither of (1.38)–(1.39) is summable for any $n \geq 2$, nor are (1.27) or (1.29) summable for $p = 2$. The failure of summability of (1.27)–(1.28) for $p = 1$ accords with the divergence of the susceptibility at the critical point.

Notable features of our method of proof are that:

- (i) The case of $n = 0$ is united with the case $n \geq 1$ despite the apparent differences in the definitions of the WSAW and $|\varphi|^4$ models.
- (ii) The proof proceeds via second-order perturbative calculations [13] of the sort found in non-rigorous renormalisation group calculations in the physics literature, but here with all error terms rigorously controlled via a general renormalisation group method [28, 29].
- (iii) There is a different renormalisation group flow due to the reduced $O(n)$ symmetry in the proof of (1.38)–(1.39), compared to the $O(n)$ symmetric case of (1.40). This is the origin of the different powers in the logarithmic corrections.

First steps towards the application of the method to critical correlation functions were made in [11], where the case $n = 0$, $p = 1$ was studied. Here we significantly extend the methods applied in [11] to obtain a much more general result, which identifies logarithmic corrections that appear when $p \geq 2$ and reveals new phenomena for the case $n \geq 2$, $p = 2$.

For the continuous-time weakly self-avoiding walk on a 4-dimensional *hierarchical* lattice, much more has been proved [18, 23, 24, 61]; in particular, the asymptotic behaviour of the end-to-end distance is identified in [23]. The decay of the analogue of $\langle |\varphi_a|^2; |\varphi_b|^2 \rangle_{\nu_c}$ below the critical dimension has been studied rigorously in a hierarchical setting of quantum fields over the p -adics [1].

2 Reformulation of the problem

Initially, the definitions of the $|\varphi|^4$ and WSAW models appear quite different. In this section, we develop a unified formulation of the problems addressed in our main theorems. We begin in Sections 2.1–2.2 by recalling and extending the connection between the $|\varphi|^4$ and WSAW models, which arises from an integral representation of WSAW. Such integral representations are discussed at length in [25]. Using the integral representation, the WSAW star and watermelon networks are expressed in terms of functional integrals which involve a complex boson field ϕ and a fermion field ψ , with quartic self-interaction. The renormalisation group method we apply is well suited to the analysis of such problems with or without the fermion field, and both models can be handled together, once we replace the Gaussian expectation for the $|\varphi|^4$ model by a Gaussian super-expectation, as discussed in Section 2.3. The specific correlation functions studied in our main theorems are obtained via the use of *observable* and *external* fields, which we introduce in Section 2.4–2.5. There we reformulate the basic problem in a unified manner for both models in terms of these auxiliary fields.

2.1 Infinite volume limit for WSAW

The integral representation for WSAW is for a finite volume version, and we first show how the watermelon and star networks on \mathbb{Z}^d can be approximated by networks on a torus. Let E_a^N denote the expectation corresponding to p independent continuous-time simple random walks on the torus Λ_N , started at $a \in \Lambda_N$. Let $b \in \Lambda_N$. For $p \geq 1$, we define a finite volume version of the p -watermelon (1.17), by

$$W_{ab,N}^{(p)}(g, \nu) = p! \int_{\mathbb{R}_+^p} E_a^N [e^{-gI_p(T)} \mathbb{1}_{X(T)=b}] e^{-\nu \|T\|_1} dT, \quad (2.1)$$

and a finite volume version of the star network (1.18), by

$$S_N^{(p)}(g, \nu) = p! \sum_{b_1, \dots, b_p \in \Lambda_N} \int_{\mathbb{R}_+^p} E_a^N [e^{-gI_p(T)} \mathbb{1}_{X^1(T)=b_1} \cdots \mathbb{1}_{X^p(T)=b_p}] e^{-\nu \|T\|_1} dT \quad (2.2)$$

(which is independent of a by translation invariance).

By the argument under (1.17), $W_{ab,N}^{(p)} \leq p!(W_{ab,N}^{(1)})^p$. By the Cauchy–Schwarz inequality, $T_1 = \sum_{x \in \Lambda} L_{T_1}(x) \leq (|\Lambda| I_1(T_1))^{1/2}$, so $I_1(T_1) \geq T_1^2 / |\Lambda|$, from which we conclude that $W_{ab,N}^{(1)}$, and hence $W_{ab,N}^{(p)}$, is finite for all $g > 0$ and $\nu \in \mathbb{R}$. Similarly, $S_N^{(p)}(g, \nu) < \infty$ for all $g > 0$ and $\nu \in \mathbb{R}$.

Proposition 2.1. *For $d > 0$ and $g > 0$,*

$$W_{ab}^{(p)}(g, \nu_c) = \lim_{\nu \downarrow \nu_c} W_{ab}^{(p)}(g, \nu) = \lim_{\nu \downarrow \nu_c} \lim_{N \rightarrow \infty} W_{ab,N}^{(p)}(g, \nu), \quad (2.3)$$

and, for $\nu \in \mathbb{R}$,

$$S^{(p)}(g, \nu) = \lim_{N \rightarrow \infty} S_N^{(p)}(g, \nu). \quad (2.4)$$

Proof. The first equality in (2.3) holds by monotone convergence. An elementary proof of the second equality is given in [11, Proposition 2.1] for the case of $p = 1$. That proof generalises directly to the case of $p \geq 1$, and we omit the details.

The proof of (2.4) for general $p \geq 1$ is a straightforward generalisation of the proof for $p = 1$ given in [12, Lemma 2.1], and again we omit the details. Both sides of (2.4) are finite for $\nu > \nu_c$, but the proof gives equality also when both sides are infinite. \blacksquare

2.2 Integral representation for WSAW

Let $M = |\Lambda_N| = L^{Nd}$. Let $u_1, v_1, \dots, u_M, v_M$ be standard coordinates on \mathbb{R}^{2M} . Then $du_1 \wedge dv_1 \wedge \dots \wedge du_M \wedge dv_M$ is the standard volume form on \mathbb{R}^{2M} , where \wedge denotes the anticommuting wedge product. The one-forms du_x, dv_y generate the Grassmann algebra of differential forms on \mathbb{R}^{2M} . We multiply differential forms using the wedge product, but for notational simplicity we do not display the wedge explicitly, and write, e.g., $du_x dv_y$ in place of $du_x \wedge dv_y$. The order of differentials in a product therefore matters. A form given by a function of u, v times a product of p differentials, or a sum of such, is said to have *degree* p , for $p \geq 0$. A form which is a sum of forms of even degree is called *even*. The integral of a differential form over \mathbb{R}^{2M} is defined to be zero unless the form has degree $2M$. A form of degree $2M$ can be written as $K = f(u, v) du_1 dv_1 \dots du_M dv_M$, and we define

$$\int K = \int_{\mathbb{R}^{2M}} f(u, v) du_1 dv_1 \dots du_M dv_M, \quad (2.5)$$

where the right-hand side is the Lebesgue integral of f over \mathbb{R}^{2M} .

We set $\phi_x = u_x + iv_x$, $\bar{\phi}_x = u_x - iv_x$ and $d\phi_x = du_x + idv_x$, $d\bar{\phi}_x = du_x - idv_x$, for $x \in \Lambda$. Since the wedge product is anticommutative, the following pairs all anticommute for every $x, y \in \Lambda$: $d\phi_x$ and $d\phi_y$, $d\bar{\phi}_x$ and $d\bar{\phi}_y$, $d\phi_x$ and $d\bar{\phi}_x$. Also,

$$d\bar{\phi}_x d\phi_x = 2i du_x dv_x. \quad (2.6)$$

The integral $\int f(\phi, \bar{\phi}) \prod_{x \in \Lambda} d\bar{\phi}_x d\phi_x$ is thus given by $(2i)^M$ times the Lebesgue integral of $f(u + iv, u - iv)$ over \mathbb{R}^{2M} . The product over x can be taken in any order, since each factor $d\bar{\phi}_x d\phi_x$ has even degree. We write

$$\psi_x = \frac{1}{(2\pi i)^{1/2}} d\phi_x, \quad \bar{\psi}_x = \frac{1}{(2\pi i)^{1/2}} d\bar{\phi}_x, \quad (2.7)$$

with a fixed choice of the square root. Then

$$\bar{\psi}_x \psi_x = \frac{1}{2\pi i} d\bar{\phi}_x d\phi_x = \frac{1}{\pi} du_x dv_x. \quad (2.8)$$

We refer to $\phi, \bar{\phi}$ as the *boson* field and to $\psi, \bar{\psi}$ as the *fermion* field.

Let \mathcal{N}^\otimes denote the algebra of even differential forms. An element $K \in \mathcal{N}^\otimes$ can be written as

$$K = \sum_{k=0}^{2M} \sum_{p, q: p+q=2k} \sum_{x_1, \dots, x_p \in \Lambda} \sum_{y_1, \dots, y_q \in \Lambda} K_{x, y} \psi^x \bar{\psi}^y, \quad (2.9)$$

where $x = (x_1, \dots, x_p)$, $y = (y_1, \dots, y_q)$, $\psi^x = \psi_{x_1} \cdots \psi_{x_p}$, $\bar{\psi}^y = \bar{\psi}_{y_1} \cdots \bar{\psi}_{y_q}$, and where each $K_{x,y}$ (including the degenerate case $p = q = 0$) is a function of $(\phi, \bar{\phi})$. We fix a positive integer $p_{\mathcal{N}} \geq \max\{10, 2p+4\}$ and impose the smoothness condition that elements of $\mathcal{N}^{\mathcal{O}}$ are such that the coefficients $K_{x,y}$ are in $C^{p_{\mathcal{N}}}$ (the reason for this particular choice of $p_{\mathcal{N}}$ is discussed in Section 7.1.3.)

Given a finite index set J , let $K = (K_j)_{j \in J}$ with each $K_j \in \mathcal{N}^{\mathcal{O}}$. Let K_j^0 denote the degree-zero part of K_j . Given a C^∞ function $F : \mathbb{R}^J \rightarrow \mathbb{C}$, we define $F(K)$ by its power series about the degree-zero part of K , i.e.,

$$F(K) = \sum_{\alpha} \frac{1}{\alpha!} F^{(\alpha)}(K^0) (K - K^0)^{\alpha}. \quad (2.10)$$

Here α is a multi-index, with $\alpha! = \prod_{j \in J} \alpha_j!$, and $(K - K^0)^{\alpha} = \prod_{j \in J} (K_j - K_j^0)^{\alpha_j}$. The summation terminates as soon as $\sum_{j \in J} \alpha_j = M$ since higher-order forms must vanish, and the order of the product on the right-hand side does not matter since each K_j is assumed to be even.

For $x \in \Lambda$, we define the differential forms

$$\tau_x = \phi_x \bar{\phi}_x + \psi_x \bar{\psi}_x, \quad (2.11)$$

$$\tau_{\Delta,x} = \frac{1}{2} \left(\phi_x (-\Delta \bar{\phi})_x + (-\Delta \phi)_x \bar{\phi}_x + \psi_x (-\Delta \bar{\psi})_x + (-\Delta \psi)_x \bar{\psi}_x \right), \quad (2.12)$$

where $\Delta = \Delta_{\Lambda}$ is the lattice Laplacian as defined above (1.1). The following theorem is a minor extension of [25, Theorem 5.1]; we provide a self-contained proof in Appendix A. The integrand on the left-hand side of (2.13) is defined as in (2.10), e.g., $e^{-\tau_x} = e^{-|\phi_x|^2} (1 + \psi_x \bar{\psi}_x)$, the integral is as in (2.5). On the right-hand side, S_p denotes the set of permutations of $1, \dots, p$.

Proposition 2.2. *For $d > 0$, $g > 0$, $\nu \in \mathbb{R}$, $p \geq 1$, and $A = (a_1, \dots, a_p)$, $B = (b_1, \dots, b_p)$ with each $a_i, b_j \in \Lambda_N$,*

$$\int e^{-\sum_{x \in \Lambda} (\tau_{\Delta,x} + g\tau_x^2 + \nu\tau_x)} \bar{\phi}_{a_1} \cdots \bar{\phi}_{a_p} \phi_{b_1} \cdots \phi_{b_p} = \sum_{\sigma \in S_p} \int_{\mathbb{R}_+^p} E_A^N [e^{-gI_p(T)} \mathbb{1}_{X(T)=\sigma(B)}] e^{-\nu\|T\|_1} dT, \quad (2.13)$$

where on the right-hand side $X^i(0) = a_i$ and $X^i(T_i) = \sigma(b_i)$.

Corollary 2.3. *For $d > 0$, $g > 0$, $\nu \in \mathbb{R}$, $p \geq 1$, and $a, b \in \Lambda_N$,*

$$S_N^{(p)}(g, \nu) = \sum_{b_1, \dots, b_p \in \Lambda_N} \int e^{-\sum_{x \in \Lambda_N} (\tau_{\Delta,x} + g\tau_x^2 + \nu\tau_x)} \bar{\phi}_a^p \phi_{b_1} \cdots \phi_{b_p}, \quad (2.14)$$

$$W_{ab,N}^{(p)}(g, \nu) = \int e^{-\sum_{x \in \Lambda_N} (\tau_{\Delta,x} + g\tau_x^2 + \nu\tau_x)} \bar{\phi}_a^p \phi_b^p. \quad (2.15)$$

Proof. This is an immediate consequence of Proposition 2.2 and the definitions of $S_N^{(p)}$, $W_{ab,N}^{(p)}$. ■

2.3 Change of variables and Gaussian approximation

To unify the treatment of the $|\varphi|^4$ and WSAW models, for the $|\varphi|^4$ model instead of (2.11)–(2.12) we define

$$\tau_x = \frac{1}{2}|\varphi_x|^2, \quad \tau_x^2 = \frac{1}{4}|\varphi_x|^4, \quad \tau_{\Delta,x} = \frac{1}{2}\varphi_x \cdot (-\Delta\varphi)_x. \quad (2.16)$$

For either model, given $g, \nu, z \in \mathbb{R}$, we write

$$U_{g,\nu,z;x} = g\tau_x^2 + \nu\tau_x + z\tau_{\Delta,x}. \quad (2.17)$$

The polynomial $U_{g,\nu,1;x}$ appears in (1.2) with τ and τ_{Δ} interpreted as in (2.16), and it appears in the right-hand sides of (2.14) and (2.15) with the interpretation (2.11)–(2.12). Given $X \subset \Lambda$ and $g_0, \nu_0, z_0 \in \mathbb{R}$, we define

$$U_0(X) = \sum_{x \in X} U_{g_0, \nu_0, z_0; x} = \sum_{x \in X} (g_0\tau_x^2 + \nu_0\tau_x + z_0\tau_{\Delta,x}). \quad (2.18)$$

To write our principal quantities as perturbations of a Gaussian, we make an appropriate change of variables. For $|\varphi|^4$, given $z_0 > -1$ and $m^2 > 0$, by definition,

$$U_{g,\nu,1;x}(\varphi) = U_{0,m^2,1;x}((1+z_0)^{-1/2}\varphi) + U_{g_0,\nu_0,z_0;x}((1+z_0)^{-1/2}\varphi), \quad (2.19)$$

with

$$g_0 = g(1+z_0)^2, \quad \nu_0 = (1+z_0)\nu - m^2. \quad (2.20)$$

The equations (2.20) can equivalently be written as

$$g = \frac{g_0}{(1+z_0)^2}, \quad \nu = \frac{\nu_0 + m^2}{1+z_0}. \quad (2.21)$$

For the moment, we regard m^2, z_0 as parameters that can be chosen arbitrarily. In Section 4, we make careful choices of these, corresponding to “physical mass” and “wave function renormalisation” in the physics literature. Let $C = (-\Delta + m^2)^{-1}$, with Δ the discrete Laplacian on Λ_N (acting on scalar functions). For $|\varphi|^4$, the Gaussian expectation with covariance C is defined by

$$\mathbb{E}_C F = \langle F \rangle_{0,m^2,N}. \quad (2.22)$$

Given a function $F(\varphi)$ we write $F'(\varphi) = F((1+z_0)^{1/2}\varphi)$. Using (2.19) and the change of variables $\varphi_x \mapsto \varphi' = (1+z_0)^{1/2}\varphi_x$, we obtain

$$\langle F \rangle_{g,\nu,N} = \frac{\mathbb{E}_C F' e^{-U_0(\Lambda)}}{\mathbb{E}_C e^{-U_0(\Lambda)}}. \quad (2.23)$$

For WSAW, we use the Gaussian *super-expectation*

$$\mathbb{E}_C F = \int F e^{-\sum_{x \in \Lambda} (\tau_{\Delta,x} + m^2 \tau_x)}, \quad (2.24)$$

defined for $F \in \mathcal{N}^\varnothing$ such that the integral exists. Such integrals are discussed at length for our context in [25, 26]. By Corollary 2.3 and an analogue of (2.19),

$$W_{ab,N}^{(p)}(g, \nu) = (1+z_0)^p \mathbb{E}_C (e^{-U_0(\Lambda)} \bar{\phi}_a^p \phi_b^p). \quad (2.25)$$

Unlike in (2.23), there is no division by a partition function. In fact, as a result of *supersymmetry* (see [25]), here $\mathbb{E}_C e^{-U_0(\Lambda)} = 1$. In addition, $\mathbb{E}_C (e^{-U_0(\Lambda)} \bar{\phi}_a^p) = \mathbb{E}_C (e^{-U_0(\Lambda)} \phi_b^p) = 0$, and there is no subtracted term in (2.25) like there is in the truncated correlation (1.7) for the $|\varphi|^4$ model.

2.4 Observable field

As is often the case in statistical mechanics, we compute correlation functions as derivatives with respect to an external field, which we refer to as an *observable* field. We do this in a manner similar to what is done in [11] for the case $(n, p) = (0, 1)$.

2.4.1 Observable field for $|\varphi|^4$

Given $n \geq 1$, let $S_{ij} = \langle (\varphi_a^i)^p; (\varphi_b^j)^p \rangle_{g, \nu, N}$, which is what we wish to compute. This defines a symmetric $n \times n$ matrix whose diagonal elements are the same, and whose off-diagonal elements are also the same.

We use the notation φ_x^p , which is equal to φ_x when $p = 1$, and to the vector whose components are $(\varphi_x^i)^2$ for $p = 2$. Recall the definition of U_0 in (2.18). Given a vector $h \in \mathbb{R}^n$, and given *observable fields* $\sigma_a, \sigma_b \in \mathbb{R}$, we define V_0 (which depends on h, n, p) by

$$V_{0;x} = U_{0;x} - \sigma_a(\varphi_a^p \cdot h) - \sigma_b(\varphi_b^p \cdot h). \quad (2.26)$$

Let D_{σ_a} denote the operator $\frac{\partial}{\partial \sigma_a}$ at $\sigma_a = \sigma_b = 0$, and similarly for higher derivatives. By (2.23) and calculation of the derivative,

$$h \cdot Sh = \langle \varphi_a^p \cdot h; \varphi_b^p \cdot h \rangle_{g, \nu, N} = (1 + z_0)^p D_{\sigma_a \sigma_b}^2 \log \mathbb{E}_C e^{-V_0(\Lambda)}. \quad (2.27)$$

Given the values of $h \cdot Sh$ for two choices of h , the matrix elements of S can be computed easily.

We define

$$\mathcal{N}^\varnothing = \mathcal{N}^\varnothing(\Lambda) = C^{p_N}((\mathbb{R}^n)^\Lambda, \mathbb{R}) \quad (2.28)$$

to be the space of real-valued functions of the fields having at least p_N continuous derivatives, where p_N is fixed as in Section 2.2. This is the space of random variables of initial interest, but because of the introduction of the observable fields, we are interested in functions not only of $\varphi \in (\mathbb{R}^n)^\Lambda$ but also of σ_a, σ_b . On the other hand, our ultimate interest in the dependence on the observable fields is the computation of the derivative appearing in (2.27). For this, we have no need to examine any dependence on σ_a, σ_b beyond terms of the form $1, \sigma_a, \sigma_b, \sigma_a \sigma_b$. We formalise this via the introduction of a quotient space, in which two functions of $\varphi, \sigma_a, \sigma_b$ become equivalent if their formal power series in the observable fields agree to order 1, $\sigma_a, \sigma_b, \sigma_a \sigma_b$, as follows.

Let $\tilde{\mathcal{N}}$ be the space of real-valued functions of $\varphi, \sigma_a, \sigma_b$ which are C^{p_N} in φ and C^∞ in σ_a, σ_b . Consider the elements of $\tilde{\mathcal{N}}$ whose formal power series expansion to second-order in the external field is zero. These elements form an ideal \mathcal{I} in $\tilde{\mathcal{N}}$, and the quotient algebra $\mathcal{N} = \tilde{\mathcal{N}}/\mathcal{I}$ has a direct sum decomposition

$$\mathcal{N} = \tilde{\mathcal{N}}/\mathcal{I} = \mathcal{N}^\varnothing \oplus \mathcal{N}^a \oplus \mathcal{N}^b \oplus \mathcal{N}^{ab}. \quad (2.29)$$

The elements of $\mathcal{N}^a, \mathcal{N}^b, \mathcal{N}^{ab}$ are given by elements of \mathcal{N}^\varnothing multiplied by σ_a , by σ_b , and by $\sigma_a \sigma_b$ respectively. As functions of the observable field, elements of \mathcal{N} are then identified with polynomials of degree at most 2. For example, we identify $e^{(\varphi_a \cdot h)\sigma_a + (\varphi_b \cdot h)\sigma_b}$ and $1 + (\varphi_a \cdot h)\sigma_a + (\varphi_b \cdot h)\sigma_b + (\varphi_a \cdot h)(\varphi_b \cdot h)\sigma_a \sigma_b$, as both are elements of the same equivalence class in the quotient space. An element $F \in \mathcal{N}$ can be written as

$$F = F_\varnothing + \sigma_a F_a + \sigma_b F_b + \sigma_a \sigma_b F_{ab}, \quad (2.30)$$

where $F_\alpha \in \mathcal{N}^\varnothing$ for each $\alpha \in \{\varnothing, a, b, ab\}$. We define projections $\pi_\alpha : \mathcal{N} \rightarrow \mathcal{N}^\alpha$ by $\pi_\varnothing F = F_\varnothing$, $\pi_a F = \sigma_a F_a$, $\pi_b F = \sigma_b F_b$, and $\pi_{ab} F = \sigma_a \sigma_b F_{ab}$.

2.4.2 Observable field for WSAW

For WSAW, we introduce *observable fields* $\sigma_a, \sigma_b \in \mathbb{C}$, and we extend (2.9) by now allowing the coefficients $K_{x,y}$ to be functions of σ_a, σ_b as well as of the boson field $\phi, \bar{\phi}$. Let $\tilde{\mathcal{N}}$ be the resulting algebra of differential forms, and let \mathcal{I} denote the ideal in $\tilde{\mathcal{N}}$ consisting of those elements of $\tilde{\mathcal{N}}$ whose formal power series expansion to second-order in the external field is zero. The quotient algebra $\mathcal{N} = \tilde{\mathcal{N}}/\mathcal{I}$ again has the direct sum decomposition

$$\mathcal{N} = \tilde{\mathcal{N}}/\mathcal{I} = \mathcal{N}^\varnothing \oplus \mathcal{N}^a \oplus \mathcal{N}^b \oplus \mathcal{N}^{ab}, \quad (2.31)$$

where elements of $\mathcal{N}^a, \mathcal{N}^b, \mathcal{N}^{ab}$ are respectively given by elements of \mathcal{N}^\varnothing multiplied by σ_a , by σ_b , and by $\sigma_a \sigma_b$. For example, $\phi_x \bar{\phi}_y \psi_x \bar{\psi}_x \in \mathcal{N}^\varnothing$, and $\sigma_a \bar{\phi}_x \in \mathcal{N}^a$. As functions of the external field, elements of \mathcal{N} are again identified with polynomials in the external fields with terms of order 1, $\sigma_a, \sigma_b, \sigma_a \sigma_b$. We use canonical projections π_α also for WSAW, as defined below (2.30).

Let

$$V_{0;x} = U_{0;x} - \sigma_a \bar{\phi}_a^p \mathbb{1}_{x=a} - \sigma_b \phi_b^p \mathbb{1}_{x=b}. \quad (2.32)$$

Then the expectation $\mathbb{E}_C e^{-V_0(\Lambda)}$ is well-defined for any $p \geq 1$, including large p , since the superficially dangerous factor $\exp[\sigma_a \bar{\phi}_a^p + \sigma_b \phi_b^p]$ is equivalent to a polynomial in the fields, which is integrable (see Section 7.1.5). With this interpretation, for all $p \geq 1$,

$$W_{ab,N}^{(p)}(g, \nu) = (1 + z_0)^p D_{\sigma_a \sigma_b}^2 \mathbb{E}_C e^{-V_0(\Lambda)}. \quad (2.33)$$

In view of the observations below (2.25), we may equivalently write

$$W_{ab,N}^{(p)}(g, \nu) = (1 + z_0)^p D_{\sigma_a \sigma_b}^2 \log \mathbb{E}_C e^{-V_0(\Lambda)}, \quad (2.34)$$

which has the same form as (2.27).

2.5 External field

For the proof of Theorem 1.1, we introduce a different kind of external field, as follows.

2.5.1 External field for $|\varphi|^4$

Let $n \geq 1$. We refer to a function $J : \Lambda \rightarrow \mathbb{R}^n$ as an *external field*. We define the inner product (\cdot, \cdot) of two fields as $(\varphi, J) = \sum_{x \in \Lambda} \varphi_x \cdot J_x$, where $\varphi_x \cdot J_x$ is the standard dot product on \mathbb{R}^n . We typically write $H : \Lambda \rightarrow \mathbb{R}^n$ for a *constant* field, with $H_x = H_0$ for every $x \in \Lambda$. For k a non-negative integer, we write $D_J^k(H)$ for the operation of k^{th} directional derivative with respect to J at $J = 0$, with each derivative taken in direction H . Then by (2.23) and direct computation of the derivative, for $p = 1, 2$,

$$\langle (\varphi, H)^p; \varphi_a^p \cdot h \rangle_{g,\nu,N} = (1 + z_0)^p D_J^p(H) D_{\sigma_a} \log \mathbb{E}_C e^{-V_0(\Lambda) + (\varphi, J)}. \quad (2.35)$$

Finite volume correlations as in (1.21)–(1.23) can be written in the form (2.35) with appropriate choices of $H, h \in \mathbb{R}^n$.

2.5.2 External field for WSAW

For WSAW, we use conjugate external fields $J, \bar{J} : \Lambda \rightarrow \mathbb{C}$. Let 1 denote the constant test function $1_x = 1$ for all $x \in \Lambda$. We define D_J^k to be the operator of k directional derivatives with respect to \bar{J} in the direction 1 at $(J, \bar{J}) = (0, 0)$, i.e., $D_J^k F(J, \bar{J}) = \frac{\partial}{\partial s_1}|_0 \cdots \frac{\partial}{\partial s_k}|_0 F(0, 0 + s_1 1 + \cdots + s_k 1)$. Direct computation gives

$$S_N^{(p)}(g, \nu) = (1 + z_0)^p D_J^p D_{\sigma_a} \mathbb{E}_C e^{-V_0(\Lambda) + (J, \bar{\phi}) + (\bar{J}, \phi)}. \quad (2.36)$$

3 Perturbative renormalisation group flow

In Section 3.1, we recall how a Gaussian expectation (or super-expectation) can be evaluated progressively in an iterative fashion. This provides the basis for the renormalisation group approach. In Section 3.2, we identify a class of local field polynomials that is important for our analysis, and recall the projection operator Loc from [27], which projects \mathcal{N} onto local polynomials. In Section 3.3 we recall from [28] the definition of a replacement $I(V, \Lambda)$ for $e^{-V(\Lambda)}$, which is better suited to the renormalisation group iteration. We also define the perturbative flow of coupling constants used in the proof of Theorem 1.2. Finally, in Section 3.4, we perform an explicit computation of the perturbative flow for observables, in Proposition 3.2, which provides the basis for the computation of the logarithmic powers in our main results.

3.1 Progressive Gaussian integration

We call $C = (-\Delta_{\Lambda_N} + m^2)^{-1}$ the *covariance*. According to (2.27) and (2.34), our goal is the computation of the expectation

$$\mathbb{E}_C e^{-V_0(\Lambda)}. \quad (3.1)$$

For $n \geq 1$, V_0 is given by (2.26) and the expectation is the standard Gaussian expectation (2.22). For $n = 0$, V_0 is given by (2.32) and the expectation is the Gaussian super-expectation (2.24). We compute these expectations *progressively*, using covariance decomposition.

We use decompositions of the two covariances $(-\Delta_{\mathbb{Z}^d} + m^2)^{-1}$ and $(-\Delta_{\Lambda_N} + m^2)^{-1}$. For \mathbb{Z}^d , the covariance exists for $d > 2$ for all $m^2 \geq 0$, but for Λ_N we must restrict to $m^2 > 0$ since the finite-volume Laplacian is not invertible. In [13, Section 6.1], results from [10, 21] are applied to define a sequence $(C_j)_{1 \leq j < \infty}$ (depending on $m^2 \geq 0$) of positive definite covariances on \mathbb{Z}^d such that

$$(\Delta_{\mathbb{Z}^d} + m^2)^{-1} = \sum_{j=1}^{\infty} C_j \quad (m^2 \geq 0). \quad (3.2)$$

For $j \geq 0$, we define the partial sums

$$w_j = \sum_{i=1}^j C_i, \quad w_0 = 0. \quad (3.3)$$

The covariances C_j are translation invariant, and have the *finite-range* property

$$C_{j;xy} = 0 \quad \text{if} \quad |x - y| \geq \frac{1}{2} L^j. \quad (3.4)$$

For $j < N$, the covariances C_j can therefore be identified with covariances on $\Lambda = \Lambda_N$, and we use both interpretations. For $m^2 > 0$, there is also a covariance $C_{N,N}$ on Λ such that

$$(-\Delta_{\Lambda_N} + m^2)^{-1} = \sum_{j=1}^{N-1} C_j + C_{N,N}. \quad (3.5)$$

Good estimates on C_j and $C_{N,N}$ are given in [13, Proposition 6.1].

For $n \geq 1$, we write $\mathbb{E}_C \theta F$ for the convolution of F with the Gaussian measure, i.e., given an integrable $F \in \mathcal{N}$, we define

$$(\mathbb{E}_C \theta F)(\varphi) = \mathbb{E}_C F(\varphi + \zeta), \quad (3.6)$$

where the expectation \mathbb{E}_C acts on ζ and leaves φ fixed. It is thus a conditional expectation.

For the case of WSAW, given an additional boson field $\xi, \bar{\xi}$ and an additional fermion field $\eta, \bar{\eta}$, with $\eta = \frac{1}{\sqrt{2\pi i}} d\xi$, $\bar{\eta} = \frac{1}{\sqrt{2\pi i}} d\bar{\xi}$, we consider the ‘‘doubled’’ algebra $\mathcal{N}(\Lambda \sqcup \Lambda')$ containing the original fields and also these additional fields. We define a map $\theta : \mathcal{N}(\Lambda) \rightarrow \mathcal{N}(\Lambda \sqcup \Lambda')$ by making the replacement in an element of \mathcal{N} of ϕ by $\phi + \xi$, $\bar{\phi}$ by $\bar{\phi} + \bar{\xi}$, ψ by $\psi + \eta$, and $\bar{\psi}$ by $\bar{\psi} + \bar{\xi}$. Then for $F \in \mathcal{N}(\Lambda)$, $\mathbb{E}_C \theta F$ is obtained by regarding the expectation as an integral over the variables $\xi, \bar{\xi}, \eta, \bar{\eta}$ which leaves the variables $\phi, \bar{\phi}, \psi, \bar{\psi}$ fixed.

According to [26, Proposition 2.6], for both WSAW and $n \geq 1$ we have

$$\mathbb{E}_C \theta F = (\mathbb{E}_{C_{N,N}} \theta \circ \mathbb{E}_{C_{N-1}} \theta \circ \cdots \circ \mathbb{E}_{C_1} \theta) F. \quad (3.7)$$

This expresses the expectation on the left-hand side as a progressive integration. To compute the expectation $\mathbb{E}_C e^{-V_0(\Lambda)}$ of (3.1), we use (3.7) to evaluate the more general quantity $\mathbb{E}_C \theta e^{-V_0(\Lambda)}$ progressively. Namely, we define

$$Z_{j+1} = \mathbb{E}_{C_{j+1}} \theta Z_j \quad (0 \leq j < N), \quad (3.8)$$

with $Z_0 = e^{-V_0(\Lambda)}$, and with an abuse of notation in that we interpret C_N as $C_{N,N}$. By (3.7), we can evaluate $\mathbb{E}_C F$ by setting the fields equal to zero in

$$Z_N = \mathbb{E}_C \theta Z_0. \quad (3.9)$$

Thus we are led to study the recursion $Z_j \mapsto Z_{j+1}$. We write $\mathbb{E}_j = \mathbb{E}_{C_j}$, and leave implicit the dependence of the covariance C_j on the mass m .

Given a *scale* $j \in \{0, 1, \dots, N\}$, we partition $\Lambda = \mathbb{Z}^d / L^N \mathbb{Z}^d$ into a disjoint union of $L^{d(N-j)}$ *scale- j blocks* of side length L^j , and denote the set of all such blocks by \mathcal{B}_j . One block contains the origin at its corner and is of the form $\{x \in \Lambda : |x|_\infty < L^j\}$, and all other blocks are translates of this one by vectors in $L^j \mathbb{Z}^d$. A *scale- j polymer* is a union of scale- j blocks, and we write $\mathcal{P}_j = \mathcal{P}_j(\Lambda)$ for the set of scale- j polymers. Given $a, b \in \Lambda$, an important scale is the *coalescence scale* j_{ab} , defined by

$$j_{ab} = \lfloor \log_L(2|a - b|) \rfloor. \quad (3.10)$$

Thus j_{ab} is the unique integer such that

$$\frac{1}{2} L^{j_{ab}} \leq |a - b| < \frac{1}{2} L^{j_{ab}+1}. \quad (3.11)$$

By (3.4), the smallest j for which $C_{j;ab} \neq 0$ is possible is $j = j_{ab} + 1$.

3.2 Field polynomials and the operator Loc

3.2.1 Approximation via cumulant expansion

To illustrate the ideas involved in the study of the recursion $Z_j \mapsto Z_{j+1}$, we consider the computation of $Z_1 = \mathbb{E}_1 \theta e^{-V_0(\Lambda)}$, at the level of formal power series accurate to second order in the coupling constants of V_0 . This can be done by expansion of $e^{-V_0(\Lambda)}$ to second order, and the result can be written

$$Z_1 = \mathbb{E}_{C_1} \theta e^{-V_0} \approx \exp \left(-\mathbb{E}_{C_1} \theta V_0 + \frac{1}{2} \mathbb{E}_{C_1} \theta (V_0; V_0) \right), \quad (3.12)$$

where $\mathbb{E} \theta (V_0; V_0) = \mathbb{E} \theta V_0^2 - (\mathbb{E} \theta V_0)^2$, and \approx denotes approximation accurate to second order in the sense of formal power series. This is an instance of the *cumulant expansion*. Then

$$Z_1 \approx e^{-H_1} \quad \text{with} \quad H_1 = \mathbb{E}_{C_1} \theta V_0 - \frac{1}{2} \mathbb{E}_{C_1} \theta (V_0; V_0). \quad (3.13)$$

The polynomial H_1 can be computed explicitly, as follows. We define operators

$$\mathcal{L}_C = \frac{1}{2} \sum_{u,v \in \Lambda} C_{u,v} \sum_{i=1}^n \frac{\partial}{\partial \varphi_u^i} \frac{\partial}{\partial \varphi_v^i}, \quad \mathcal{L}_C = \sum_{u,v \in \Lambda} C_{u,v} \left(\frac{\partial}{\partial \phi_u} \frac{\partial}{\partial \phi_v} + \frac{\partial}{\partial \psi_u} \frac{\partial}{\partial \psi_v} \right), \quad (3.14)$$

for the $|\varphi|^4$ and WSAW models, respectively. Then, for a polynomial A in the fields,

$$\mathbb{E}_C \theta A = e^{\mathcal{L}_C} A, \quad (3.15)$$

where the exponential on the right-hand side is defined by its power series expansion (a finite series when applied to a polynomial); see [26, Lemma 4.2] for a proof. The computation of the first term $\mathbb{E}_{C_1} \theta V_0$ of H_1 in (3.13) is elementary; details for observables are given in Section 3.4 below. The second term $\mathbb{E}_{C_1} \theta (V_0; V_0)$ is bilinear and can be computed as a sum of terms arising from the monomials in V_0 . For the case $(n, p) = (1, 1)$, using (3.15) we find that one of these terms is

$$\mathbb{E}_{C_1} \theta (\sigma_a \varphi_a; \sigma_b \varphi_b) = \sigma_a \sigma_b (\varphi_a \varphi_b + C_{1;ab}) - \sigma_a \sigma_b \varphi_a \varphi_b = \sigma_a \sigma_b C_{1;ab}. \quad (3.16)$$

There is no $\sigma_a \sigma_b$ term in V_0 , and the creation of such a term in H_1 is welcome, as the second derivatives on the right-hand sides of (2.27) and (2.33) would produce a non-zero result when applied to e^{-H_1} but not to e^{-V_0} .

One lesson learned from the above computation is that expectation can create new terms that did not appear in V_0 , such as $\sigma_a \sigma_b C_{1;ab}$. To accommodate this, we will define an n -dependent class of polynomials \mathcal{V}_h that is stable under the action of the progressive integration, to second-order approximation as above.

A second lesson from the above computation is that not all terms in H_1 are *local*, due to the nonlocal nature of the operator \mathcal{L}_C in (3.14). To deal with this issue, we use the projection operator Loc of [27], which we discuss in Section 3.2.3.

3.2.2 Local field polynomials

Given $h \in \mathbb{R}^n$, $h \neq 0$, we define a class of local polynomials \mathcal{V}_h that can be used to parametrise the result of progressive expectations. It is necessary to keep track of the dependence on the vector h

for $n \geq 2$, whereas for $n = 0$ and $n = 1$ we simply set $h = 1$. We define

$$\rho_x^a(h) = \begin{cases} \bar{\phi}_x^p & (n = 0) \\ (\varphi_x^p \cdot h) / |h| & (n \geq 1), \end{cases} \quad \rho_x^b(h) = \begin{cases} \phi_x^p & (n = 0) \\ (\varphi_x^p \cdot h) / |h| & (n \geq 1). \end{cases} \quad (3.17)$$

Note that ρ_x^α depends on the vector h only through its direction. We also need the monomial

$$\tau_{\nabla\nabla,x} = \begin{cases} \frac{1}{2} \sum_{e \in \mathbb{Z}^d: |e|=1} ((\nabla^e \phi)_x (\nabla^e \bar{\phi})_x + (\nabla^e \psi)_x (\nabla^e \bar{\psi})_x) & (n = 0) \\ \frac{1}{4} \sum_{e \in \mathbb{Z}^d: |e|=1} \nabla^e \varphi_x \cdot \nabla^e \varphi_x & (n \geq 1). \end{cases} \quad (3.18)$$

Then we define the polynomials (for $\alpha = a, b$)

$$V_{\emptyset,x} = g\tau_x^2 + \nu\tau_x + z\tau_{\Delta,x} + y\tau_{\nabla\nabla,x} + u, \quad V_{\alpha,x} = \begin{cases} \lambda_\alpha \rho_x^\alpha(h) & (n = 0) \\ \lambda_\alpha \rho_x^\alpha(h) + t_\alpha & (n \geq 1), \end{cases} \quad (3.19)$$

and define

$$\mathcal{V}_h = \{V : V_x = V_{\emptyset,x} - \sigma_a V_{a,x} \mathbb{1}_{x=a} - \sigma_b V_{b,x} \mathbb{1}_{x=b} - \sigma_a \sigma_b \frac{1}{2} (q_a \mathbb{1}_{x=a} + q_b \mathbb{1}_{x=b})\}. \quad (3.20)$$

Given $X \subset \Lambda$, we also define

$$\mathcal{V}_h(X) = \{V(X) = \sum_{x \in X} V_x : V \in \mathcal{V}_h\}. \quad (3.21)$$

The scalar coefficients in the above polynomials are all real numbers for the $|\varphi|^4$ model. For the WSAW, all are real except $\lambda_a, \lambda_b, q_a, q_b$ which are permitted to be complex (this is discussed further in Section 7.1.1 below).

Two useful subspaces of \mathcal{V}_h are the subspace $\mathcal{V}_h^{(0)}$ consisting of elements of \mathcal{V}_h with $u = y = t_a = t_b = q_a = q_b = 0$, and the subspace $\mathcal{V}_h^{(1)}$ consisting of elements with $y = 0$. The polynomial V_0 of (3.1) lies in the subset of $\mathcal{V}_h^{(0)}$ with $\lambda_a = \lambda_b = 1$.

3.2.3 Localisation

Let $X \subset \Lambda$. We now recall some basics about the localisation operator $\text{Loc}_X : \mathcal{N} \rightarrow \mathcal{V}(X)$, which projects \mathcal{N} onto a vector space $\mathcal{V}(X)$ of local polynomials that in general contains more monomials than $\mathcal{V}_h(X)$. The definition and general theory of this operator is given in [27], and we adapt the theory here to incorporate the observables.

By definition, the operator Loc_X respects the direct sum decomposition (2.29)–(2.31) of \mathcal{N} , in the sense that $\text{Loc}_X \pi_\alpha = \pi_\alpha \text{Loc}_X$ for each $\alpha = \{\emptyset, a, b, ab\}$. We omit discussion of a detail that limits the domain of \mathcal{N} to avoid issues associated with “wrapping around” the torus Λ , this point is discussed carefully in [27]. The restrictions $\text{Loc}_X|_{\mathcal{N}^\alpha}$ are defined individually for each α . As discussed in detail in [27], their definitions require: (i) specification of the dimensions of the fields, (ii) choice of a maximal monomial dimension $d_+(\alpha)$ for each α , and (iii) choice of covariant field polynomials \hat{P} which form the basis for the vector space $\text{Loc}_X(\mathcal{N}^\alpha)$ (see [27, Definition 1.2]). Item (iii) is done exactly as in [27, (1.19)]; this item does not play any significant role in the present paper and we will not discuss it further. Also, since we do not make explicit use of $\text{Loc}_X|_{\mathcal{N}^\emptyset}$ in this paper, we do not specify its definition in detail, which is identical to what is used in [12, 15]. For the observable components of Loc_X , we use the following.

(i) The *dimensions* of the fields are simply

$$[\varphi] = 1, \quad [\phi] = [\bar{\phi}] = [\psi] = [\bar{\psi}] = 1. \quad (3.22)$$

By definition, the dimension of a monomial $\nabla^\eta \zeta$ is $|\eta|_1 + [\zeta] = |\eta|_1 + 1$, where η is a multi-index and ζ may be any of φ or $\phi, \bar{\phi}, \psi, \bar{\psi}$. Here $\nabla^\eta = \nabla_1^{\eta_1} \cdots \nabla_d^{\eta_d}$ for a multi-index $\eta = (\eta_1, \dots, \eta_d)$, where ∇_k denotes the finite-difference operator $(\nabla_k f)_x = f_{x+e_k} - f_x$, with e_k the k^{th} standard unit vector. The dimension of a product of such monomials is the sum of the dimensions of the factors in the product. This is the same as in [12, 15].

(ii) For $\alpha = ab$, we set $d_+(ab) = 0$. For $\alpha = a$ and $\alpha = b$, we make the scale dependent choice $d_+^{(j)}(a) = d_+^{(j)}(b) = p \mathbb{1}_{j < j_{ab}}$. The reduction of $d_+^{(j)}(a)$ at the coalescence scale j_{ab} (defined in (3.10)) is a decision that simplifies some aspects in Section 5 below; this decision was also taken in [11].

3.3 Definition of I_j

We now recall several definitions which are made and explained in [13, 28]. We use the direct sum decompositions $\mathcal{N} = \mathcal{N}^\emptyset \oplus \mathcal{N}^a \oplus \mathcal{N}^b \oplus \mathcal{N}^{ab}$ of (2.29)–(2.31), the canonical projections π_α as defined under (2.30), and the abbreviation $\pi_* = 1 - \pi_\emptyset = \pi_a + \pi_b + \pi_{ab}$.

For polynomials A, B in the fields, and with \mathcal{L}_C given by (3.14), we define

$$F_C(A, B) = e^{\mathcal{L}_C} (e^{-\mathcal{L}_C} A) (e^{-\mathcal{L}_C} B) - AB, \quad (3.23)$$

$$F_{\pi, C}(A, B) = F_C(A, \pi_\emptyset B) + F_C(\pi_* A, B). \quad (3.24)$$

Recall that the covariance w_j is defined by (3.3). For a local polynomial V in the fields, and for a polymer $X \in \mathcal{P}_j$, we set

$$W_j(V, X) = \frac{1}{2} \sum_{x \in X} (1 - \text{Loc}_x^{(j)}) F_{\pi, w_j}(V_x, V(\Lambda)). \quad (3.25)$$

(The definition (3.25) is inapplicable for the final scale $j = N$; this special case is discussed in [28, Section 1.1.5].) Then, for $X \in \mathcal{P}_j$, we define the *interaction* functional

$$I_j(V, X) = e^{-V(X)} \prod_{B \in \mathcal{B}_j(X)} (1 + W_j(V, B)). \quad (3.26)$$

For $j = 0$, where $w_0 = 0$, we interpret the above as $I_0(V, X) = e^{-V(X)}$.

Let $\mathcal{L}_{j+1} = \mathcal{L}_{C_{j+1}}$. Given V , we define

$$P_{j,x} = \frac{1}{2} \sum_{y \in \Lambda} (\text{Loc}_x^{(j+1)} F_{\pi, w_{j+1}}(e^{\mathcal{L}_{j+1}} V_x, e^{\mathcal{L}_{j+1}} V_y) - e^{\mathcal{L}_{j+1}} \text{Loc}_x^{(j+1)} F_{\pi, w_j}(V_x, V_y)), \quad (3.27)$$

and set

$$V_{\text{pt}, j+1, x}(V) = e^{\mathcal{L}_{j+1}} V_x - P_{j,x}. \quad (3.28)$$

The definition (3.28) is equivalent to the definition in [13, (3.23)], by [13, Lemmas 5.5–5.6]. The definition is motivated in [13], where it is shown that

$$\mathbb{E}_{j+1} \theta I_j(V, \Lambda) \approx I_{j+1}(V_{\text{pt}}, \Lambda), \quad (3.29)$$

with “ \approx ” as in Section 3.2 above. Equation (3.29) shows that, to second order, I enjoys a form of stability under expectation when V is advanced to V_{pt} . However, at this point there is no uniformity in scale j or volume Λ in the error estimate.

3.4 Perturbative flow of coupling constants

The perturbative flow of the bulk coupling constants g, ν, z, y is given in [13] for WSAW and in [15] for $|\varphi|^4$. In [13] it is also given for the observable coupling constants λ_α, q_α for WSAW, for the specific case $p = 1$. In the following proposition, we extend the perturbative computation to the observables needed for our main results, and compute $\pi_* V_{\text{pt}}$ for all $p \geq 1$ for WSAW, and for $p = 1, 2$ for $|\varphi|^4$. For this, we need some preliminaries.

Definition 3.1. For $n \geq 2$, we write $M_2(n)$ for the set of $n \times n$ matrices of the form $rI + sJ$, with $r, s \in \mathbb{R}$, I the identity matrix, and J having all entries equal to 1.

The vector

$$e^+ = (1, 1, \dots, 1) \in \mathbb{R}^n \quad (3.30)$$

appears frequently in our analysis. Every matrix in $M_2(n)$ has eigenspaces E^\pm , where $E^+ = \text{span}(e^+)$ with eigenvalue $r + ns$, and E^- is the orthogonal complement $E^- = (E^+)^\perp$ with eigenvalue r .

Let I denote $1 \in \mathbb{R}$ when $n = 0$ and the $n \times n$ identity matrix for $n \geq 1$. We define a matrix T , which is in $M_2(n)$ for $n \geq 2$, by

$$T = \begin{cases} \binom{p}{2} \frac{1}{4} I & (n = 0) \\ \binom{p}{2} I \frac{1}{3} & (n = 1) \\ \binom{p}{2} \frac{2}{n+8} I + \binom{p}{2} \frac{1}{n+8} J & (n \geq 2). \end{cases} \quad (3.31)$$

The matrix T is the zero matrix for $p = 1$ (as $\binom{1}{2} = 0$), and otherwise has eigenspace E^+ with eigenvalue $\gamma_{n,p}^+ = \binom{p}{2} \frac{n+2}{n+8}$, and for $n \geq 2$ also has eigenspace $E^- = (E^+)^\perp$ with eigenvalue $\gamma_{n,p}^- = \binom{p}{2} \frac{2}{n+8}$. The correspondence between the matrix T for $n \geq 1$ and the value we have assigned to $n = 0$ should be understood via the eigenvalues, as $\frac{0+2}{0+8} = \frac{2}{0+8} = \frac{1}{4}$. For $n = 0, 1$ there is only γ^+ and E^+ . For $n \geq 2$ and $p = 2$, we have $\gamma^- = \frac{2}{n+8} < \frac{n+2}{n+8} = \gamma^+ < 1$, and this is the only setting where both eigenvalues play a role in our analysis.

For $q : \Lambda \rightarrow \mathbb{R}$, let $q^{(n)} = \sum_{x \in \Lambda} q_{0,x}^n$. Let $C = C_{j+1}$, $w = w_j$, and, for $g, \nu \in \mathbb{R}$, let

$$\nu^+ = \nu + g(n+2)C_{00}, \quad \delta_j[f(\nu, w)] = f(\nu^+, w + C) - f(\nu, w), \quad (3.32)$$

$$\beta_j = (n+8)\delta_j[w^{(2)}]. \quad (3.33)$$

We define the matrix

$$A_j = \begin{cases} (1 - p\delta_j[\nu w^{(1)}])I - \beta_j g T & (j+1 < j_{ab}) \\ I & (j+1 \geq j_{ab}). \end{cases} \quad (3.34)$$

Thus A_j is $n \times n$ for $n \geq 1$ and 1×1 for $n = 0$. For $n \geq 2$, $A_j \in M_2(n)$. The eigenvalues and eigenvectors of A_j play an important role in identifying the logarithmic corrections in Theorem 1.2. The eigenspaces are E^\pm , with eigenvalues

$$f_j^\pm = \begin{cases} 1 - p\delta_j[\nu w^{(1)}] - \beta_j g \gamma_{n,p}^\pm & (j+1 < j_{ab}) \\ 1 & (j+1 \geq j_{ab}). \end{cases} \quad (3.35)$$

The following proposition computes V_{pt} as a function of V . For its statement, we define

$$\varsigma_j = C_{0,0}(1 - \mathbb{1}_{j+1 < j_{ab}} 2\nu w^{(1)}) + \mathbb{1}_{j+1 < j_{ab}} \nu^+ \delta_j[w^{(2)}] + \mathbb{1}_{j+1 \geq j_{ab}} \delta_j[\nu w^{(2)}]. \quad (3.36)$$

Proposition 3.2. *Let $d = 4$. Let $p \geq 1$ for WSAW, and $p = 1, 2$ for $|\varphi|^4$. Let $V \in \mathcal{V}_h$ with $|h| = 1$. Then $V_{\text{pt},j+1}(V) \in \mathcal{V}_{h_{\text{pt}}}$, and for $x = a, b$, $\pi_* V_{\text{pt},j+1}$ is given by*

$$h_{\text{pt}} = (A_j h) / |A_j h|, \quad (3.37)$$

$$\lambda_{\text{pt},x} = |A_j h| \lambda_x, \quad (3.38)$$

$$q_{\text{pt},x} = q_x + p! \lambda_a \lambda_b \delta_j[w_{ab}^p], \quad (3.39)$$

$$t_{\text{pt},x} = t_x + \mathbb{1}_{n \geq 1} \mathbb{1}_{p=2} \lambda_x (e^+ \cdot h) \varsigma_j. \quad (3.40)$$

In particular, if $h \in E^\pm$, then $h_{\text{pt}} = h$ and $\lambda_{\text{pt},x} h_{\text{pt}} = f_j^\pm \lambda_x h$.

It is clear from (3.37) that for $V \in \mathcal{V}_h$ it is in general not the case that V_{pt} lies in \mathcal{V}_h when $n \geq 2$. Instead, $V_{\text{pt}} \in \mathcal{V}_{h_{\text{pt}}}$ for a new direction h_{pt} . However, if h is in one of the eigenspaces E^\pm , then $h_{\text{pt}} = h$. To have $h_{\text{pt}} = h$ is a desirable simplification, and this gives the eigenspaces E^\pm a special significance.

The proof of Proposition 3.2 involves similar but not identical calculations for $n = 0$ and $n \geq 1$. However, once Proposition 3.2 is proved, the remaining analysis for the proof of our main results is unified for all $n \geq 0$.

As noted below (3.11), $j = j_{ab}$ is the smallest scale j for which $C_{j+1,ab} \neq 0$ is possible, and so $\delta_i[w_{ab}^p]$ can be nonzero for the first time also when $i = j_{ab}$. Therefore the first scale for which $q_{\text{pt}} - q$ can be nonzero is $q_{\text{pt},j_{ab}+1}$.

For the rest of this section, we write $w = w_j$, $C = C_{j+1}$ and $\mathcal{L} = \mathcal{L}_{C_{j+1}}$. The first step in the proof of Proposition 3.2 is the computation of the first term in $V_{\text{pt}} = e^{\mathcal{L}} V - P$ of (3.28), provided by the following lemma.

Lemma 3.3. *Let $n = 0$ and $p \geq 1$, or let $n \geq 1$ and $p = 1, 2$. For $V \in \mathcal{V}_h$,*

$$e^{\mathcal{L}} V_x = V_x + g(n+2) C_{00} \tau_x + \mathbb{1}_{n \geq 1} (\delta u_{\text{pt}} - \mathbb{1}_{p=2} \lambda_x (e^+ \cdot h) C_{00} \sigma_x), \quad (3.41)$$

where δu_{pt} is an explicit quadratic function of $g, \nu, y + z$.

Proof. The computation of $e^{\mathcal{L}} \pi_\emptyset V_x$ is carried out in [12, 15] and agrees with the above formula. In particular, $\delta u_{\text{pt}} = 0$ for $n = 0$, and δu_{pt} is given by [15, (3.27)] for $n \geq 1$. For the observable part, for $n = 0$ we have $\mathcal{L} \pi_* V = 0$ and hence $\pi_* e^{\mathcal{L}} V_x = \pi_* V_x$, as in (3.41). For $n \geq 1$ and $p = 1, 2$, we have $\mathcal{L}^2 \pi_* V = 0$, so $\pi_* e^{\mathcal{L} C} V_x = \pi_* V_x + \mathcal{L} \pi_* V_x$. Direct calculation of $\mathcal{L} \pi_* V_x$ gives the final term of (3.41). \blacksquare

To compute $\pi_* P_x$ we use (3.27), i.e.,

$$\pi_* P_x = \frac{1}{2} \sum_{y \in \Lambda} \left(\text{Loc}_x \pi_* F_{\pi, w+C}(e^{\mathcal{L}} V_x, e^{\mathcal{L}} V_y) - e^{\mathcal{L}} \text{Loc}_x \pi_* F_{\pi, w}(V_x, V_y) \right), \quad (3.42)$$

in conjunction with (3.24) which implies

$$\pi_* F_{\pi, w}(V_x, V_y) = 2F_w(\pi_* V_x, \pi_{\emptyset} V_y) + F_w(\pi_* V_x, \pi_* V_y). \quad (3.43)$$

For the following lemma, for each pair $x, y \in \Lambda$, we define an $n \times n$ matrix (1×1 if $n = 0$)

$$M_{xy} = \mathbb{1}_{j+1 < j_{ab}} (\nu p w_{xy} I + g(n+8) w_{xy}^2 T). \quad (3.44)$$

Lemma 3.4. *Let $n = 0$, $p \geq 1$, or $n \geq 1$, $p = 1, 2$. For $V \in \mathcal{V}_h$,*

$$\text{Loc}_x F_w(\pi_* V_x, \pi_{\emptyset} V_y) = -\sigma_x \lambda_x \left((M_{xy} \varphi_x^p \cdot h) + \mathbb{1}_{n \geq 1} \mathbb{1}_{p=2} \nu w_{xy}^2 (e^+ \cdot h) \right), \quad (3.45)$$

$$\text{Loc}_x F_w(\pi_* V_x, \pi_* V_y) = -\sigma_a \sigma_b p! \lambda_a \lambda_b |h|^2 w_{xy}^p (\mathbb{1}_{x=a} \mathbb{1}_{y=b} + \mathbb{1}_{x=b} \mathbb{1}_{y=a}), \quad (3.46)$$

where for $n = 0$ we interpret φ_x on the right-hand side of (3.45) as $\bar{\phi}_a$ for $x = a$ and ϕ_b for $x = b$.

Proof. We evaluate F using [13, Lemma 5.6], which implies that for $n \geq 1$,

$$F_C(A_x, B_y) = \sum_{k=1}^D \frac{1}{k!} \sum_{i_1, \dots, i_k=1}^n \sum_{\substack{u_l, v_l \in \Lambda \\ (l=1, \dots, k)}} \left(\prod_{l=1}^k C_{u_l, v_l} \right) \frac{\partial^k A_x}{\partial \varphi_{u_1}^{i_1} \dots \partial \varphi_{u_k}^{i_k}} \frac{\partial^k B_y}{\partial \varphi_{v_1}^{i_1} \dots \partial \varphi_{v_k}^{i_k}}, \quad (3.47)$$

with $D = \deg A \vee \deg B$. For $n = 0$, there is a related formula that also involves the fermions.

For (3.45), we first note that there is no contribution from the terms involving t_α or q_α in V_x , since the sum in (3.47) starts at $k = 1$ and hence always involves differentiation with respect to φ , which is absent in these terms. The cases π_a and π_b are symmetric, and we therefore only consider π_a . It can be argued on the basis of dimensional considerations that there is no contribution due to the terms $y\tau_{\nabla\nabla} + z\tau_{\Delta}$ in $\pi_{\emptyset} V$. For the remaining calculation, we use the notation appropriate for $n \geq 1$ and comment on what is different for $n = 0$. To prove (3.45), we therefore compute

$$\text{Loc}_a F_w(\pi_a V_a, \pi_{\emptyset} V_y) = -\lambda_a \sum_{i=1}^n h^i \left(g \text{Loc}_x F_w(\sigma_a(\varphi_a^i)^p, \tau_y^2) + \nu \text{Loc}_x F_w(\sigma_a(\varphi_a^i)^p, \tau_y) \right). \quad (3.48)$$

For $n \geq 1$ and $p = 1, 2$,

$$F_w((\varphi_x^i)^p, |\varphi_y|^2) = F_w((\varphi_x^i)^p, (\varphi_y^i)^2) = 2p w_{xy} (\varphi_x^i)^{p-1} (\varphi_y^i) + 2 \binom{p}{2} w_{xy}^2, \quad (3.49)$$

while for $n = 0$ and $p \geq 1$,

$$F_w(\bar{\phi}_x^p, |\phi_y|^2) = p w_{xy} \bar{\phi}_x^{p-1} \bar{\phi}_y. \quad (3.50)$$

Thus, for all (n, p) under consideration,

$$\text{Loc}_a F_w(\sigma_a(\varphi_a^i)^p, \tau_y) = \sigma_a \left(\mathbb{1}_{j+1 < j_{ab}} p w_{xy} (\varphi_a^i)^p + \mathbb{1}_{n \geq 1} \mathbb{1}_{p=2} w_{xy}^2 \right), \quad (3.51)$$

with the modification noted below (3.46) for $n = 0$. For $n \geq 1$ and $p = 1, 2$,

$$F_w((\varphi_x^i)^p, |\varphi_y|^4) = F_w((\varphi_x^i)^p, (\varphi_y^i)^4) + 2F_w((\varphi_x^i)^p, (\varphi_y^i)^2) \sum_{j:j \neq i} (\varphi_y^j)^2, \quad (3.52)$$

$$F_w((\varphi_x^i)^p, (\varphi_y^i)^4) = 4pw_{xy}(\varphi_x^i)^{p-1}(\varphi_y^i)^3 + 12\binom{p}{2}w_{xy}^2(\varphi_y^i)^2, \quad (3.53)$$

while for $n = 0$ and $p \geq 1$,

$$F_w(\bar{\phi}_x^p, |\phi_y|^4) = 2pw_{xy}\bar{\phi}_x^{p-1}\phi_y\bar{\phi}_y^2 + 2\binom{p}{2}w_{xy}^2\bar{\phi}_x^{p-2}\bar{\phi}_y^2. \quad (3.54)$$

The terms of total degree above p are annihilated by Loc , and

$$\text{Loc}_a \left[F_w(\sigma_a(\varphi_x^i)^p, (\varphi_y^i)^2) \sum_{j \neq i} (\varphi_y^j)^2 \right] = \sigma_a \mathbb{1}_{n \geq 1} 2\binom{p}{2}w_{xy}^2 \sum_{j:j \neq i} (\varphi_y^j)^2. \quad (3.55)$$

Thus, for all (n, p) under consideration, we have

$$\text{Loc}_a F_w(\sigma_a(\varphi_x^i)^p, \tau_y^2) = \sigma_a \mathbb{1}_{j+1 < j_{ab}}(n+8)w_{xy}^2 (T\varphi_a^p)^i. \quad (3.56)$$

Assembly of the above completes the proof of (3.45). We omit the simpler proof of (3.46). \blacksquare

Proof of Proposition 3.2. Equation (3.39) states that

$$q_{\text{pt},x} = q_x + p!\lambda_a\lambda_b\delta[w_{ab}^p], \quad (3.57)$$

and this is an immediate consequence of (3.41), (3.42) and (3.46). To prove (3.37)–(3.38), we use

$$\lambda_{\text{pt},x}h_{\text{pt}} = \lambda_x h - \mathbb{1}_{j+1 < j_{ab}}(p\delta[\nu w^{(1)}]I - \beta gT)\lambda_x h. \quad (3.58)$$

The first term on the right-hand side arises from (3.41), and the rest of the right-hand side of (3.58) arises from P_a , via (3.42) and the first term on the right-hand side of (3.45) (using also Lemma 3.3).

Finally, we prove (3.40). The t_x term in $e^{\mathcal{L}}V$ is equal to $-t_x - \mathbb{1}_{n \geq 1}\mathbb{1}_{p=2}C_{00}\lambda_x(e^+ \cdot h)$, by Lemma 3.3. The contribution due to $-P$ arises only from the first term on the right-hand side of (3.43), and only for $n \geq 1$ and $p = 2$, by Lemma 3.4. Thus we seek the contribution to $t_{\text{pt},x}$ due to

$$-\sum_{y \in \Lambda} (\text{Loc}_x F_{w+C}(e^{\mathcal{L}}V_x, e^{\mathcal{L}}V_y) - e^{\mathcal{L}}\text{Loc}_x F_w(V_x, V_y)). \quad (3.59)$$

We apply Lemma 3.3 and (3.45) to see that the first term contributes a t_x -term which is equal to $\mathbb{1}_{n \geq 1}\mathbb{1}_{p=2}\nu^+ w_{j+1}^{(2)}\lambda_x(e^+ \cdot h)$, and the second contributes

$$-\sum_{y \in \Lambda} \lambda_x(M_{xy}e^{\mathcal{L}}\varphi_x^p \cdot h) - \mathbb{1}_{n \geq 1}\mathbb{1}_{p=2}\nu w^{(2)}. \quad (3.60)$$

The latter produces a t_x -term

$$- \mathbb{1}_{n \geq 1} \mathbb{1}_{p=2} \left(\mathbb{1}_{j+1 < j_{ab}} [p\nu w^{(1)} + g(n+2)w^{(2)}] C_{00} + \nu w^{(2)} \right) \lambda_x(e^+ \cdot h), \quad (3.61)$$

where we have used $Te^+ = \frac{n+2}{n+8}e^+$ for $p = 2$. This leads to

$$t_{\text{pt},x} = t_x + \mathbb{1}_{n \geq 1} \mathbb{1}_{p=2} \lambda_x(e^+ \cdot h) \left(C_{0,0} + \delta[\nu w^{(2)}] \right. \\ \left. - \mathbb{1}_{j+1 < j_{ab}} C_{0,0} [2\nu w^{(1)} + g(n+2)w^{(2)}] \right), \quad (3.62)$$

which is equivalent to (3.40) by definition of ν^+ and ς . This completes the proof. \blacksquare

4 Non-perturbative renormalisation group coordinate

Proposition 3.2 gives the evolution of the observable coupling constants, as defined by the map $V \mapsto V_{\text{pt}}$. As discussed around (3.29), this map describes the effect of taking the expectation at a single scale, but only at a perturbative level. In this section, we present aspects of the formalism of [12, 29], which introduces and employs a non-perturbative renormalisation group coordinate K . With this coordinate, Proposition 3.2 can be supplemented so as to obtain a rigorous non-perturbative analysis, including observables. A new ingredient is required here to deal with observables when $n \geq 2$, namely the notion of h -factorisability which is defined in Section 4.2 and developed further in Section 7.3.

4.1 Circle product

Recall that the sets \mathcal{B}_j and \mathcal{P}_j of scale- j blocks and polymers in Λ are defined in Section 3.1. For maps $F, G : \mathcal{P}_j \rightarrow \mathcal{N}$, we define the *circle product* $F \circ G : \mathcal{P}_j \rightarrow \mathcal{N}$ by

$$(F \circ G)(X) = \sum_{Y \in \mathcal{P}_j(X)} F(X \setminus Y) G(Y) \quad (X \in \mathcal{P}_j). \quad (4.1)$$

The empty set \emptyset is a polymer, as is Λ , so the sum over Y always includes $Y = \emptyset$, and includes $Y = \Lambda$ when $X = \Lambda$. Every map $F : \mathcal{P}_j \rightarrow \mathcal{N}$ that we encounter obeys $F(\emptyset) = 1$. The circle product is commutative and associative, and has unit element $\mathbb{1}_\emptyset$ defined by $\mathbb{1}_\emptyset(X) = 1$ if $X = \emptyset$ and otherwise $\mathbb{1}_\emptyset(X) = 0$.

We define

$$I_0(X) = e^{-V_0(X)}, \quad K_0(X) = \mathbb{1}_\emptyset(X). \quad (4.2)$$

Then

$$Z_0 = e^{-V_0(\Lambda)} = I_0(\Lambda) = (I_0 \circ K_0)(\Lambda). \quad (4.3)$$

We wish to maintain the form of (4.3) after each expectation in the progressive expectation (3.7). Namely, we seek to define polynomials $U_j \in \mathcal{V}_h^{(0)}$, constants $u_j, t_{a,j}, t_{b,j}, q_{a,j}, q_{b,j}$, and a *non-perturbative coordinate* $K_j : \mathcal{P}_j \rightarrow \mathcal{N}_j$, such that Z_j of (3.8) is given by

$$Z_j = e^{\zeta_j} (I_j \circ K_j)(\Lambda), \quad \zeta_j = -u_j |\Lambda| + (t_{a,j} \sigma_a + t_{b,j} \sigma_b) + \frac{1}{2} (q_{a,j} + q_{b,j}) \sigma_a \sigma_b, \quad (4.4)$$

with $I_j = I_j(U_j)$ given by (3.26). We systematically use the symbol U for elements of $\mathcal{V}_h^{(0)}$ and V for other polynomials. Let $\delta\zeta_{j+1} = \zeta_{j+1} - \zeta_j$. Then (3.8) can equivalently be written as

$$\mathbb{E}_{j+1}\theta(I_j \circ K_j)(\Lambda) = e^{-\delta\zeta_{j+1}}(I_{j+1} \circ K_{j+1})(\Lambda). \quad (4.5)$$

By the definition in (3.9), $Z_N = Z_N(\varphi) = (\mathbb{E}_C\theta Z_0)(\varphi)$ depends on the field φ . Omission of θ in (3.1) corresponds to the evaluation at $\varphi = 0$. Since, $I_j(0) = 1$, at the final scale, where $\mathcal{P}_N = \{\emptyset, \Lambda_N\}$,

$$\mathbb{E}_C Z_0 = Z_N(0) = e^{\zeta_N}(1 + K_N(\Lambda; 0)). \quad (4.6)$$

To prove Theorem 1.2, our goal is to achieve (4.6) with $K_N(\Lambda; 0)$ an error term, so that $\mathbb{E}_C Z_0$ is to leading order equal to e^{ζ_N} . Assuming this, we can evaluate the watermelon network (2.34) or the correlation function (2.27) by computing the derivative $D_{\sigma_a\sigma_b}^2 \zeta_N = \frac{1}{2}(q_{a,N} + q_{b,N})$. Thus, for both models, the important information for Theorem 1.2 is ultimately encoded in $q_{x,N}$. Similarly, we will show in Section 6, that the important information for Theorem 1.1 is encoded in $\lambda_{x,N}$.

4.2 Symmetries and symmetry reduction

New considerations concerning symmetry, not present in [28, 29] are needed for our analysis of observables when $n \geq 2$. We present the relevant definitions here.

Definition 4.1. *Lattice symmetry.* Let \mathcal{A} denote the set of graph automorphisms of Λ , i.e., bijections that preserve nearest neighbours. An automorphism $A \in \mathcal{A}$ acts on \mathcal{N} via $AF(\varphi) = F(A\varphi)$, where $(A\varphi)_x = \varphi_{Ax}$. We say that $F \in \mathcal{N}$ is *Euclidean invariant* if $AF = F$ for all $A \in \mathcal{A}$. We say that a function $F : \mathcal{P}_j \rightarrow \mathcal{N}$ is *Euclidean covariant* if $A(F(X)) = F(AX)$ for all automorphisms A of Λ and all $X \in \mathcal{P}_j$.

Definition 4.2. *Field symmetry.* For $n \geq 1$, an $n \times n$ real matrix m acts on $F \in \mathcal{N}$ via $(mF)(\varphi) = F(m\varphi)$. There is no action of m on σ_a or σ_b . Given a matrix group G , we say that $F \in \mathcal{N}$ is *G-invariant* if $mF = F$ for all $m \in G$.

For $n = 0$, let $G = U(1)$ be the group $\{z \in \mathbb{C} : |z| = 1\}$ with complex multiplication. We set $\sigma_a = \sigma^p$ and $\sigma_b = \bar{\sigma}^p$, with $\sigma \in \mathbb{C}$. Then $m \in U(1)$ acts on $F \in \mathcal{N}$ by $(mF)(\sigma, \bar{\sigma}, \phi, \bar{\phi}, \psi, \bar{\psi}) = F(m\sigma, \bar{m}\bar{\sigma}, m\phi, \bar{m}\bar{\phi}, m\psi, \bar{m}\bar{\psi})$. We say that F is *U(1)-invariant*, or *gauge invariant*, if $mF = F$ for all $m \in U(1)$.

The supersymmetry operator Q is defined, e.g., in [13, Section 5] or [25, Section 6], and we say that F is *supersymmetric* if $QF = 0$. Supersymmetry is special to the $n = 0$ case and does not play a role for observables; the rest of this paper can be read without delving into its precise meaning.

By definition, for $n = 0$, elements of \mathcal{V}_h of (3.20) are $U(1)$ -invariant. For $n \geq 1$, we use the following matrix groups:

- ▶ $G = O(n)$, the group of $n \times n$ orthogonal matrices.
- ▶ $G = S(n)$, the permutation subgroup of $O(n)$, consisting of the $n!$ matrices obtained by permutations of the columns of the identity matrix.
- ▶ $G = R(n)$, the reflection subgroup of $O(n)$ consisting of the 2^n diagonal matrices with diagonal elements in $\{-1, +1\}$.

Although $O(n)$ -invariance will hold for the *bulk* space \mathcal{N}^\emptyset for all $n \geq 1$, for $n \geq 2$ the $O(n)$ symmetry can be reduced by choice of h . This can be seen already from the $\varphi_a^p \cdot h$ term in $V_{0,x}$, which is not $R(n)$ invariant when $p = 1$, and which is not $S(n)$ -invariant for $p = 2$ unless h is in the eigenspace E^+ spanned by $e^+ = (1, \dots, 1)$. We now define a weaker property that replaces $O(n)$ -invariance for the observable terms when $n \geq 2$, and that plays a role in the definition of the Banach space in which the non-perturbative coordinate K lies.

Definition 4.3. Let $n \geq 1$ and fix $h \in \mathbb{R}^n$. We say that $F \in \mathcal{N}$ is *h -factorisable* if for $\alpha = a, b$:

- (i) there exists $F_\alpha^* \in (\pi_\emptyset \mathcal{N})^n$ (depending on h , not unique) such that $\pi_\alpha F = \sigma_\alpha(F_\alpha^* \cdot h)$, and
- (ii) $(PF_\alpha^*)(\varphi) = F_\alpha^*(P\varphi)$ for all $P \in S(n)$, where by definition PF_α^* is the result of permuting the components of F_α^* with the permutation P .

We write $\mathcal{N}_{h\text{-fac}} = \{F \in \mathcal{N} : F \text{ is } h\text{-factorisable}\}$ for the vector space of h -factorisable elements of \mathcal{N} .

In the following definition, h does not play a direct role as a vector when $n = 0, 1$ but we nevertheless use it as a notational device to write \mathcal{N}_h as the vector subspace of \mathcal{N} that obeys the conditions listed in the definition. For $n = 0$, we say that F *has no constant part* if its degree-zero part (as a form) is equal to zero when evaluated at $\phi = \bar{\phi} = 0$.

Definition 4.4. For $n \geq 0$, let \mathcal{N}_h denote the subspace of all $F \in \mathcal{N}$ such that

- (i) $\pi_\emptyset F$ is Euclidean invariant.
- (ii) If $n = 0$, $\pi_\emptyset F$ is supersymmetric, F is $U(1)$ -invariant, and F has no constant part.
- (iii) If $n \geq 1$, $F \in \mathcal{N}_{h\text{-fac}}$, $\pi_\emptyset F$ is $O(n)$ -invariant, and if in addition $p = 2$, then F is $R(n)$ -invariant.

By Proposition 3.2, if we choose $h \in E^\pm$, then $V_{\text{pt}} : \mathcal{V}_h \rightarrow \mathcal{V}_h$. The symmetry restrictions of \mathcal{N}_h , particularly h -factorisation, are used to carry this perturbative fact over to the non-perturbative renormalisation group coordinate and show that $V_j \in \mathcal{V}_h$ for all j . The two powers $\gamma_{n,p}^+$ and $\gamma_{n,p}^-$ for the logarithmic corrections in Theorems 1.1–1.2 will arise from the distinction between $h \in E^+$ and $h \in E^-$.

4.3 The non-perturbative coordinate K

We now discuss the definition of the non-perturbative coordinate K . This requires several definitions, as preparation.

A polymer $X \in \mathcal{P}_j$ is *connected* if for any $x, y \in X$ there exists a path of the form $x_0 = x, x_1, \dots, x_{n-1}, x_n = y$ with $\|x_{i+1} - x_i\|_\infty = 1$ for all i . Every polymer can be partitioned into connected components, and we denote the set of connected components of X by $\text{Comp}_j(X)$. Let $\mathcal{S}_j \subset \mathcal{P}_j$ denote the set of connected polymers consisting of at most $2^d = 16$ blocks; elements of \mathcal{S}_j are called *small sets*. (The specific number 16 plays a special role in [29], but not here.) The *small set neighbourhood* of X is

$$X^\square = \bigcup_{Y \in \mathcal{S}_j : X \cap Y \neq \emptyset} Y. \quad (4.7)$$

For $n \geq 0$, we define $\mathcal{N}(X)$ to consist of those elements of \mathcal{N} in (2.31) which depend on the boson, fermion (for $n = 0$), and external fields only at points in X , where we regard the external field σ_x as located at x for $x = a, b$. At scale j , K lies in the space \mathcal{K}_j of maps from \mathcal{P}_j to \mathcal{N} , given in the following definition.

Definition 4.5. Let $h = 1$ for $n = 0, 1$ and $h \in \mathbb{R}^n$ for $n \geq 1$. Let $\mathcal{K}_j = \mathcal{K}_j(h, \Lambda_N)$ be the vector space of functions $K : \mathcal{P}_j \rightarrow \mathcal{N}$ with the properties:

- ▶ Field locality: $K(X) \in \mathcal{N}(X^\square)$ for each connected $X \in \mathcal{P}_j$. Also, $\pi_a K(X) = 0$ if $a \notin X$, $\pi_b K(X) = 0$ if $b \notin X$, and $\pi_{ab} K(X) = 0$ if either (i) $X \in \mathcal{S}_j$ and $j < j_{ab}$ or (ii) $a \notin X$ and $b \notin X$.
- ▶ Symmetry: $K(X) \in \mathcal{N}_h$ for all $X \in \mathcal{P}_j$, except we do not assume Euclidean invariance of $K(X)$ as in Definition 4.4(i), but rather do assume Euclidean covariance of K .
- ▶ Component factorisation: $K(X) = \prod_{Y \in \text{Comp}_j(X)} K(Y)$ for all $X \in \mathcal{P}_j$.

In [29, Section 1.8], the scale dependent *renormalisation group map* from a domain in $\mathcal{V}_h^{(0)} \times \mathcal{K}_j$ to $\mathcal{V}_h^{(1)} \times \mathcal{K}_{j+1}$ is defined, which we write as

$$(U, K) \mapsto (V_+, K_+). \quad (4.8)$$

In (4.8) and elsewhere, to simplify the notation we systematically drop labels j for scale, and indicate scale $j + 1$ simply by $+$. We use the map (4.8), which satisfies (4.5). The discussion in [28, 29] is written explicitly for the WSAW with the observable having power $p = 1$, but it applies in our present more general setting with the modifications discussed in Section 7 below.

The map $(U, K) \mapsto V_+$ is explicit and relatively simple, and is defined as follows. Let $\text{Loc}_{Y,B}$ denote the operator defined by $\text{Loc}_{Y,B} F = P_Y(B)$, where P_Y is the polynomial determined by $P_Y(Y) = \text{Loc}_Y F$. We define a map $V \mapsto V^{(1)}$ from \mathcal{V}_h to $\mathcal{V}_h^{(1)}$ by replacing $z\tau_\Delta + y\tau_{\nabla\nabla}$ in $V \in \mathcal{V}_h$ by $(z + y)\tau_\Delta$ in $V^{(1)}$. We also define a map $V \mapsto V^{(0)}$ from \mathcal{V}_h to $\mathcal{V}_h^{(0)}$ by replacing $z\tau_\Delta + y\tau_{\nabla\nabla}$ in V by $(z + y)\tau_\Delta$ and replacing u, t_a, t_b, q_a, q_b in V by zero. As in [29, Section 1.8.2], the map $(U, K) \mapsto V_+$ is given by

$$V_+(U, K) = V_{\text{pt}}^{(1)}(U - Q) \quad \text{with} \quad Q(B) = \sum_{Y \in \mathcal{S}: Y \supset B} \text{Loc}_{Y,B} \left(\frac{K(Y)}{I(Y, V)} \right), \quad (4.9)$$

where V_{pt} is the explicit quadratic polynomial map $V \mapsto V_{\text{pt}}$ discussed in Section 3.4. When $K = 0$, $V_+(U, 0)$ is simply $V_{\text{pt}}^{(1)}(U)$. We write $V_+ = (\delta\zeta_+, U_+)$, and in particular $\delta\zeta_+(U, 0) = \delta\zeta_{\text{pt}}(U)$ and $U_+(U, 0) = V_{\text{pt}}^{(0)}(U)$. We express estimates on V_+ in terms of R_+ defined by

$$R_+(U, K) = V_+(U, K) - V_+(U, 0) = V_+(U, K) - V_{\text{pt}}^{(1)}(U) \in \mathcal{V}_h^{(1)}. \quad (4.10)$$

As in [29, (1.68)], the renormalisation group map has the property

$$\pi_\emptyset V_+(U, K) = V_+(\pi_\emptyset U, \pi_\emptyset K), \quad \pi_\emptyset K_+(U, K) = K_+(\pi_\emptyset U, \pi_\emptyset K). \quad (4.11)$$

Thus, under the map (4.8), the *bulk coordinates* $(\pi_\emptyset V_j, \pi_\emptyset K_j)$ satisfy a closed evolution independent of the observables. We denote this evolution map by $(V_+^\emptyset, K_+^\emptyset)$. Then the bulk part of (4.8) becomes

$$(\pi_\emptyset V_+, \pi_\emptyset K_+) = (V_+^\emptyset(\pi_\emptyset U, \pi_\emptyset K), K_+^\emptyset(\pi_\emptyset U, \pi_\emptyset K)). \quad (4.12)$$

4.4 Existence of bulk flow

A critical global renormalisation group flow of the bulk coordinates is constructed in [12] for WSAW and in [15] for $|\varphi|^4$. In particular, there is a construction of $(\pi_\emptyset V_j, \pi_\emptyset K_j)$, obeying (4.12) for all j , such that (4.5) holds if $\sigma_a = \sigma_b = 0$. The bulk flow provides detailed information about the sequence $\pi_\emptyset V_j$, and estimates on $\pi_\emptyset K_j$ sufficient for studying the infinite volume limit at the critical point.

For the bulk flow, we change perspective on which variables are independent. Both $|\varphi|^4$ and WSAW have parameters g, ν . In (2.18), additional parameters m^2, g_0, ν_0, z_0 are introduced. For the moment we consider m^2, g_0, ν_0, z_0 as four independent variables and do not work with g, ν directly. We relate m^2, g_0, ν_0, z_0 to the original parameters g, ν in Section 4.6 below.

To state the result about the bulk flow, let \bar{g}_j be the (m^2, g_0) -dependent sequence determined by $\bar{g}_{j+1} = \bar{g}_j - \beta_j \bar{g}_j^2$, with $\bar{g}_0 = g_0$, and with $\beta_j = \beta_j(m^2) = (n+8)\delta[w_j^{(2)}]$ as in (3.33). For $m^2 > 0$, we define the *mass scale* j_m to be the largest integer j such that $mL^{-j} \leq 1$, and we set $j_0 = \infty$. By definition, $\lim_{m \downarrow 0} j_m = \infty$. Given $\Omega > 1$ ($\Omega = 2$ is a good choice), we define

$$\chi_j = \Omega^{-(j-j_m)_+}, \quad (4.13)$$

where $x_+ = \max\{x, 0\}$. By [13, Lemma 6.2], $\beta_j = O(\chi_j)$ ([13, Lemma 6.2] actually shows that $\beta_j = O(\Omega^{-(j-j_\Omega)_+})$ for another scale j_Ω used in [13, 28, 29], but $\Omega^{-(j-j_\Omega)_+}$ and χ_j are comparable by [13, Proposition 4.4].) By [12, Proposition 6.1] and [12, (8.22)] respectively, the bounds

$$\chi_j \bar{g}_j^p \leq O\left(\frac{g_0}{1+g_0 j}\right)^p \quad (p \geq 0), \quad \sum_{k=j}^{\infty} \chi_k \bar{g}_k^p = O(\chi_j \bar{g}_j^{p-1}) \quad (p > 1), \quad (4.14)$$

hold uniformly in $(m^2, g_0) \in [0, \delta]^2$, for a small $\delta > 0$. The sequence \bar{g}_j converges to 0 when $m^2 = 0$ but not when $m^2 > 0$.

For WSAW, the following theorem is a consequence of [12, Proposition 8.1]. For $|\varphi|^4$, it is [15, Theorem 3.6]. The latter also controls the flow of the coupling constant u_j , which is used for the analysis of the pressure in [15] but is not needed here. The domains \mathbb{D}_j^\emptyset , and the \mathcal{W}_j -norms on the space \mathcal{K}_j , which appear in the theorem are discussed following its statement.

Theorem 4.6. *Let $d = 4$, $n \geq 0$, and let $\delta > 0$ be sufficiently small. Let $N \geq 1$. Let $(m^2, g_0) \in [0, \delta]^2$ and $\sigma_a = \sigma_b = 0$. There exist $M > 0$ and an infinite sequence of continuous functions $U_j = (g_j^c, \nu_j^c, z_j^c)$ of (m^2, g_0) , independent of the volume parameter N , such that for initial conditions $U_0 = (g_0, \nu_0^c, z_0^c)$ and $K_0 = \mathbb{1}_\emptyset$, a flow $(U_j, K_j) \in \mathbb{D}_j^\emptyset$ exists such that (4.12) holds for all $j+1 < N$, and, if $m^2 \in [\delta L^{-2(N-1)}, \delta)$, also for $j+1 = N$. Moreover, $g_j^c = O(\bar{g}_j)$, $z_j^c = O(\chi_j \bar{g}_j)$, $\nu_j = O(\chi_j L^{-2j} \bar{g}_j)$, and*

$$\|K_j\|_{\mathcal{W}_j} = \|\pi_\emptyset K_j\|_{\mathcal{W}_j} \leq M \chi_j \bar{g}_j^3 \quad (j \leq N). \quad (4.15)$$

In the remainder of the paper, we often drop the superscripts and write simply

$$U_j = (g_j, \nu_j, z_j) \quad (4.16)$$

for the sequence provided by Theorem 4.6. The stated continuity of U_j is not part of the statements of [12, Proposition 8.1] or [15, Theorem 3.6], but it is established in [12, Section 8.2].

The definition of the \mathcal{W}_j norm on \mathcal{K}_j in (4.15) is discussed at length in [29], and we do not repeat the details here. The inequality (4.15) provides various estimates on $K_j(X)$ and on its derivatives with respect to fields, in terms of the size of the polymer X . Some examples of its use are given in Lemma 6.2 below. For example, as noted explicitly in [29, (1.64)], (4.15) with $j = N$ implies that (with fields set equal to zero on the left-hand side)

$$|\pi_\emptyset K_N(\Lambda)| \leq M \chi_N \bar{g}_N^3, \quad (4.17)$$

uniformly in $m^2 \in [\delta L^{-2(N-1)}, \delta)$.

The $\mathcal{W}_j = \mathcal{W}_j(\tilde{s})$ norm depends on a parameter $\tilde{s} = (\tilde{m}^2, \tilde{g}) \in [0, \delta)^2$, whose significance is discussed in [12, Section 6.3]. Useful choices of this parameter depend on the scale j , as well as on approximate values of the mass parameter m^2 of the covariance and the coupling constant g_j . We use the convention that when the parameter \tilde{s} is omitted, it is given by $\tilde{s} = s_j = (m^2, \tilde{g}_j(m^2, g_0))$, where $\tilde{g} = \tilde{g}_j$ is defined in terms of the initial condition g_0 by

$$\tilde{g}_j = \tilde{g}_j(m^2, g_0) = \bar{g}_j(0, g_0) \mathbb{1}_{j \leq j_m} + \bar{g}_{j_m}(0, g_0) \mathbb{1}_{j > j_m}. \quad (4.18)$$

By [12, Lemma 7.4],

$$\tilde{g}_j = \bar{g}_j + O(\bar{g}_j^2), \quad (4.19)$$

so the sequences (\tilde{g}_j) and (\bar{g}_j) are the same to leading order. Moreover,

$$g_j = \bar{g}_j(1 + O(\bar{g}_j |\log \bar{g}_j|)); \quad (4.20)$$

this follows from [12, (6.1), (7.11)] for WSAW and the same result holds for $n \geq 1$ according to [15]. Thus the sequences \tilde{g}_j , \bar{g}_j and g_j are essentially interchangeable, and in particular error bounds expressed in terms of any one of them are equivalent.

The domain $\mathbb{D}_j^\emptyset = \mathbb{D}_j^\emptyset(\tilde{s}) \subset \mathcal{V}_h^\emptyset \times \mathcal{K}_j^\emptyset$ also depends on \tilde{s} (with the convention mentioned above when \tilde{s} is omitted), is independent of h as we deal only with the bulk here, and is defined as follows. For the universal constant $C_{\mathcal{D}} \geq 2$ determined in [12], for $j < N$,

$$\mathbb{D}_j^\emptyset(\tilde{s}) = \{(g, \nu, z) \in \mathbb{R}^3 : C_{\mathcal{D}}^{-1} \tilde{g} < g < C_{\mathcal{D}} \tilde{g}, L^{2j} |\nu|, |z| \leq C_{\mathcal{D}} \tilde{g}\} \times B_{\mathcal{W}_j^\emptyset}(\alpha \tilde{\chi}_j \tilde{g}^3). \quad (4.21)$$

The first factor is the stability domain defined in [28, (1.55)], restricted to the bulk coordinates and real scalars. In the second factor, $B_X(a)$ denotes the open ball of radius a centred at the origin of the Banach space X , and α is as in [12, Theorem 6.3] and [15, Theorem 3.5]; for concreteness we use $\alpha = 10M$ where M is the constant of Theorem 4.6. The space \mathcal{K}^\emptyset is the restriction of \mathcal{K} to elements K with $\pi_* K(X) = 0$ for all polymers X . Since, by (4.11), the renormalisation group acts triangularly, the distinction between \mathcal{W} and \mathcal{W}^\emptyset is unimportant for the bulk flow, and \mathcal{W}^\emptyset is denoted by \mathcal{W} in [12].

4.5 Properties of the bulk flow

We provide some details about the flow of bulk coupling constants, for later use.

The *bubble diagram* is defined by

$$\mathbf{B}_{m^2} = (n+8) \int_0^\infty \int_0^\infty P(X(T) = Y(S)) e^{-m^2 T} e^{-m^2 S} dT dS, \quad (4.22)$$

where X, Y are independent continuous-time simple random walks (taking steps at the events of a rate- $(2d)$ Poisson process). For $d = 4$, it is an exercise in calculus (see [12, (1.8)]) to see that

$$\mathbf{B}_{m^2} \sim \mathbf{b} \log m^{-2} \quad \text{as } m^2 \downarrow 0, \text{ with } \mathbf{b} = \frac{n+8}{16\pi^2}. \quad (4.23)$$

We recall from [13, Lemma 6.3] that

$$\beta_j = \mathbf{b} \log L + O(L^{-j}) \quad \text{for } m^2 = 0. \quad (4.24)$$

Lemma 4.7. *For $(m^2, g_0) \in (0, \delta)^2$, the limit $g_\infty = \lim_{j \rightarrow \infty} g_j$ exists, is continuous in (m^2, g_0) , and extends continuously to $[0, \delta)^2$. For $g_0 \in (0, \delta)$,*

$$g_\infty \sim \frac{1}{\mathbf{B}_{m^2}} \quad \text{as } m^2 \downarrow 0. \quad (4.25)$$

Proof. For $n = 0$, this is [12, Lemma 8.5], adapted from its statement for the sequence \check{g}_j to the sequence g_j . That this adaptation is possible is discussed at the end of [12, Section 8.3]. For $n \geq 1$, (4.25) also holds, as indicated in [15, (4.28)]. \blacksquare

For the next lemma, recall that $\mathcal{E}_{ab}^{(p)}$ is defined in (1.25).

Lemma 4.8. *As $|a-b| \rightarrow \infty$, $L^{j_{ab}} = 2|a-b| + O(1)$. If $j_m > j_{ab}$ then $g_{j_{ab}}^{-1} = \mathbf{b}(\log |a-b|)(1 + \mathcal{E}_{ab}^{(2)})$.*

Proof. It is an immediate consequence of (3.11) that $L^{j_{ab}} = 2|a-b| + O(1)$.

Since $j_{ab} < j_m$, we have $\tilde{g}_{j_{ab}} = \bar{g}_{j_{ab}}$ with $\bar{g}_{j_{ab}}$ defined by the sequence β_j given by $m^2 = 0$. By (4.19)–(4.20) it suffices to prove that

$$\bar{g}_{j_{ab}}(0)^{-1} = \mathbf{b}(\log |a-b|)(1 + \mathcal{E}_{ab}^{(2)}). \quad (4.26)$$

It is shown in the proof of [14, Lemma 2.1] that if $\psi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is absolutely continuous then

$$\sum_{l=j}^k \beta_l \psi(\bar{g}_l) \bar{g}_l^2 = \int_{\bar{g}_{k+1}}^{\bar{g}_j} \psi(t) dt + O\left(\int_{\bar{g}_{k+1}}^{\bar{g}_j} t^2 |\psi'(t)| dt\right). \quad (4.27)$$

Let $\beta_\infty = \mathbf{b} \log L$. We set $\psi(t) = t^{-2}$ in (4.27), and apply (4.24), to obtain

$$\bar{g}_k^{-1} = \bar{g}_0^{-1} + \sum_{j=0}^{k-1} \beta_j + O(|\log \bar{g}_k|) = \bar{g}_0^{-1} + \beta_\infty k + O(1) + O(|\log \bar{g}_k|). \quad (4.28)$$

In particular, $\bar{g}_k^{-1} = O(\bar{g}_0^{-1} + \beta_\infty k) = O(k)$ (with g_0 -dependent constant). Therefore,

$$\bar{g}_k^{-1} = \beta_\infty k + O(\log k). \quad (4.29)$$

This gives (4.26) and completes the proof. \blacksquare

Lemma 4.9. *Let $\delta_j = \delta_j[\nu w^{(1)}]$, $\delta'_j = \nu_{j+1} w_{j+1}^{(1)} - \nu_j w_j^{(1)}$. Then $\delta_j = O(\chi_j \bar{g}_j)$ and $|\delta_j - \delta'_j| = O(\chi_j \bar{g}_j^2)$.*

Proof. By [13, Lemma 6.2], $w_j^{(1)} = O(L^{2j})$. By [13, Proposition 6.1], $C_{j+1;ab} = O(\chi_j L^{-2j})$. With (3.32) and (4.21), we therefore have

$$\begin{aligned}\delta_j &= (\nu_j + (2+n)g_j C_{j+1;00})(w_j^{(1)} + C_{j+1}^{(1)}) - \nu_j w_j^{(1)} \\ &= \nu_j C_{j+1}^{(1)} + (2+n)g_j C_{j+1;00} w_{j+1}^{(1)} \\ &= O(\tilde{g}_j L^{-2j})O(\chi_j) + O(g_j)O(\chi_j L^{-2j})O(L^{2j}) = O(\chi_j \bar{g}_j).\end{aligned}\tag{4.30}$$

For the second statement, by definition

$$\delta'_j = (\nu_{j+1} - (\nu_j + (2+n)g_j C_{j+1;00})) w_{j+1}^{(1)}.\tag{4.31}$$

The subtracted terms in the difference on the right-hand side cancel the first-order part of ν_{j+1} (see [13, (3.31)]), leaving higher-order terms which are bounded by $O(\chi_j L^{-2j} \bar{g}_j^2)$ according to [29, (1.80)]. This leads to the desired bound on δ'_j . \blacksquare

Recall from (3.35) that the eigenvalues of the matrix A_j defined in (3.34) are $f_j = 1 - p\delta_j[\nu w^{(1)}] - \beta_j g_j \gamma$, where now g_j, ν_j (and also z_j) are given by the flow of the bulk coupling constants determined in Theorem 4.6. The constant γ is given by $\gamma = \gamma_{n,p}^\pm$, depending on the values of (n, p) and the choice of $h \in E^\pm$. For $j \leq J$, we write

$$\Pi_{j,J} = \prod_{i=j}^J f_i, \quad \Pi_j = \Pi_{0,j}.\tag{4.32}$$

The value of $\Pi_{j,J}$ depends on γ , and we write $\Pi_{j,J}^\pm$ for its values when $\gamma = \gamma^\pm$. The matrix product $A_J A_{J-1} \cdots A_j$ has eigenvalues $\Pi_{j,J}^\pm$, with the eigenvalue $\Pi_{j,J}^-$ only occurring for $n \geq 2$ and $p = 2$. Error estimates in the following lemma depend on γ , but this is unimportant since γ is fixed.

Lemma 4.10. *Let $(m^2, g_0) \in [0, \delta]$. Let $0 \leq j \leq J < \infty$ and $\gamma \in \mathbb{R}$. There exists $\alpha_j = 1 + O(\bar{g}_j)$ such that*

$$\Pi_{j,J} = \alpha_j \left(\frac{g_{J+1}}{g_j} \right)^\gamma (1 + O(\chi_J \bar{g}_J)).\tag{4.33}$$

Proof. For $i \in \mathbb{N}$, let $\delta_i = \delta_i[\nu w^{(1)}]$. By Theorem 4.6, the sequence g_i obeys the recursion relation

$$g_{i+1} = (1 - e_i)g_i \quad \text{with} \quad e_i = \beta_i g_i + 4\delta_i + \tilde{r}_i, \quad \tilde{r}_i = O(\chi_i \bar{g}_i^2).\tag{4.34}$$

As noted below (4.13), $\beta_j = O(\chi_j)$. By Lemma 4.9, $\delta_i = O(\chi_i \bar{g}_i)$. Therefore, $e_i = O(\chi_i \bar{g}_i)$. Let $\delta'_i = \nu_{i+1} w_{i+1}^{(1)} - \nu_i w_i^{(1)}$. By Lemma 4.9, $|\delta_i - \delta'_i| = O(\chi_i \bar{g}_i^2)$. By (3.35),

$$f_i = (1 - \gamma e_i)(1 + d_i),\tag{4.35}$$

with

$$d_i = (1 - \gamma e_i)^{-1}((4\gamma - p)\delta_i + \gamma \tilde{r}_i) = (4\gamma - p)\delta'_i + O(\chi_j \bar{g}_i^2).\tag{4.36}$$

By Taylor's theorem, for small t ,

$$1 - \gamma t = (1 - t)^\gamma (1 + O(t^2)).\tag{4.37}$$

Therefore,

$$f_i = (1 - e_i)^\gamma (1 + O(\chi_i \bar{g}_i^2))(1 + d_i) = \left(\frac{g_{i+1}}{g_i} \right)^\gamma (1 + E_i), \quad (4.38)$$

with

$$E_i = d_i + O(\chi_i \bar{g}_i^2) = (4\gamma - p)\delta'_i + O(\chi_i \bar{g}_i^2) = O(\chi_i \bar{g}_i^2). \quad (4.39)$$

Let

$$\alpha_j = \prod_{i=j}^{\infty} (1 + E_i). \quad (4.40)$$

The infinite product converges since E_i is conditionally summable and $\sum_{i=0}^{\infty} E_i^2 < \infty$. Moreover, $\alpha_j = 1 + O(\sum_{i=j}^{\infty} E_i) = 1 + O(\chi_j \bar{g}_j)$, by (4.14). With (4.32), we obtain

$$\Pi_{j,J} = \alpha_j \left(\frac{g_{J+1}}{g_j} \right)^\gamma \alpha_J^{-1} = \alpha_j \left(\frac{g_{J+1}}{g_j} \right)^\gamma (1 + O(\chi_J \bar{g}_J)), \quad (4.41)$$

and the proof is complete. \blacksquare

For $j \geq 0$, in view of Lemma 4.10 it is natural to define

$$\Gamma_j = (g_j/g_0)^\gamma. \quad (4.42)$$

Lemma 4.11. *As $|a - b| \rightarrow \infty$, if $j_m > j_{ab}$ then*

$$\Gamma_{j_{ab}} = \left(\frac{1}{\mathfrak{b}g_0 \log |a - b|} \right)^\gamma (1 + \mathcal{E}_{ab}^{(p)}). \quad (4.43)$$

Proof. Since $L^{j_{ab}} = 2|a - b| + O(1)$ by Lemma 4.8, (4.43) follows from (4.42) and Lemma 4.8. The error estimate improves for $p = 1$ because in this case $\gamma = 0$; in fact $\Gamma_j = 1$ for all j when $p = 1$. \blacksquare

4.6 Change of variables

Theorem 4.6 is stated in terms of the parameters m^2, g_0 , rather than the parameters g, ν that define the WSAW and $|\varphi|^4$ models. The following proposition, proved in [12, Proposition 4.2(ii)] for WSAW and [15, (4.23)] for $|\varphi|^4$, relates these sets of parameters via the functions z_0^c, ν_0^c of Theorem 4.6 and (2.20). The critical value ν_c enters the analysis here, for the first time.

Proposition 4.12. *Let $d = 4$, $n \geq 0$, and $\delta_1 > 0$ be small enough. There exists a function $[0, \delta_1]^2 \rightarrow [0, \delta]^2$, that we denote by $(g, \varepsilon) \mapsto (\tilde{m}^2(g, \varepsilon), \tilde{g}_0(g, \varepsilon))$, such that (2.20) holds with $\nu = \nu_c(g) + \varepsilon$, if $z_0 = z_0^c(\tilde{m}^2, \tilde{g}_0)$ and $\nu_0 = \nu_0^c(\tilde{m}^2, \tilde{g}_0)$. The functions \tilde{m}, \tilde{g}_0 are right-continuous as $\varepsilon \downarrow 0$, and satisfy $\tilde{m}^2(g, 0) = 0$, and $\tilde{m}^2(g, \varepsilon) > 0$ if $\varepsilon > 0$.*

We also define the right-continuous functions (as $\varepsilon \downarrow 0$)

$$\tilde{z}_0(g, \varepsilon) = z_0^c(\tilde{m}^2(g, \varepsilon), \tilde{g}_0(g, \varepsilon)), \quad \tilde{\nu}_0(g, \varepsilon) = \nu_0^c(\tilde{m}^2(g, \varepsilon), \tilde{g}_0(g, \varepsilon)). \quad (4.44)$$

Starting from (g, ν) , Proposition 4.12 provides $(\tilde{m}^2, \tilde{g}_0)$, and then Theorem 4.6 provides an initial condition $U_0 = (\tilde{g}_0, \tilde{z}_0, \tilde{\nu}_0)$ for which there exists a global bulk flow of the renormalisation group map. This needs to be supplemented by the observable flow, whose perturbative part is given by Proposition 3.2. In the next section, we analyse the complete renormalisation group flow, including the non-perturbative corrections for the observable flow.

5 Complete renormalisation group flow

We now augment the bulk flow provided by Theorem 4.6 to obtain a complete renormalisation group flow, including observables. In Section 5.1, we introduce the domain for the complete renormalisation group flow. The main result concerning a single renormalisation group step, Theorem 5.1, is stated in Section 5.2 with proof deferred to Section 7. In Sections 5.3–5.4, we apply Theorem 5.1 to conclude that the renormalisation group step can be iterated indefinitely. This is used in Sections 6.3–6.4 to prove our main results Theorems 1.1–1.3.

5.1 Parameters, norms and domains

We use several norms, and domains defined via these norms. The norms extend those in [29, Section 1.7] where only the two-point function was considered, to handle the new observables present here.

The following sequences \mathfrak{h}_j and $\mathfrak{h}_{\sigma,j}$ each have distinct values in two distinct cases, which we identify as either the $\mathfrak{h} = \ell$ or $\mathfrak{h} = \tilde{h}$ cases. This \tilde{h} , which is called h in [12, 15, 28, 29], is not related to and should not be confused with the vector $h \in \mathbb{R}^n$ used to define the space \mathcal{V}_h . The two options for $\mathfrak{h}_j, \mathfrak{h}_{\sigma,j}$ are used to construct the $T_{\phi,j}(\mathfrak{h}_j)$ norm in [29].

For $\ell_0, k_0 > 0$ as in [29, Section 1.7.1], and for $j \geq 0$, let

$$\mathfrak{h}_j = \begin{cases} \ell_j = \ell_0 L^{-j} & (\mathfrak{h} = \ell) \\ h_j = k_0 \tilde{g}_j^{-1/4} L^{-j} & (\mathfrak{h} = \tilde{h}). \end{cases} \quad (5.1)$$

With the notation $x \wedge y = \min\{x, y\}$ and $x_+ = \max\{x, 0\}$, we also define

$$\mathfrak{h}_{\sigma,j} = \Gamma_{j \wedge j_{ab}}^{-1} \ell_{j \wedge j_{ab}}^{-p} 2^{p(j-j_{ab})_+} \times \begin{cases} \tilde{g}_j & (\mathfrak{h} = \ell) \\ \tilde{g}_j^{p/4} & (\mathfrak{h} = \tilde{h}). \end{cases} \quad (5.2)$$

The occurrence of Γ in (5.2) is a feature that is not visible in [11], since if $p = 1$ then $\gamma = 0$ and $\Gamma = 1$. The definition here is more subtle, as it anticipates the ultimate appearance of logarithmic corrections for $p \geq 2$. It plays an important role in Lemma 6.2 below.

A j -dependent norm on \mathcal{V}_h is defined, using the $\mathfrak{h} = \ell$ case of (5.1)–(5.2), by

$$\|V\|_{\mathcal{V}_h} = \max \left\{ |g|, L^{2j} |\nu_j|, |z_j|, |y_j|, \ell_j^p \ell_{\sigma,j} (|\lambda_a| \vee |\lambda_b|), \right. \\ \left. \ell_{\sigma,j} (|t_a| \vee |t_b|), \ell_{\sigma,j}^2 (|q_a| \vee |q_b|), L^{4j} |u| \right\}, \quad (5.3)$$

where $x \vee y = \max\{x, y\}$. We extend the domain in \mathbb{R}^3 appearing in (4.21) by including now the coupling constants λ_a, λ_b (for $n = 0$ these are permitted to be complex), and define

$$\mathcal{D}_j = \{U \in \mathcal{V}_h^{(0)} : g > C_{\mathcal{D}}^{-1} \tilde{g}, \|U\|_{\mathcal{V}_h} < C_{\mathcal{D}} \tilde{g}\}. \quad (5.4)$$

The \mathcal{W} norm is built from the $T_\phi = T_{\phi,j}(\mathfrak{h}_j)$ norm used in [29], and obeys (recall (2.30))

$$\|F\|_{T_\phi} = \|F_\emptyset\|_{T_\phi} + \mathfrak{h}_\sigma \left(\|F_a\|_{T_\phi} + \|F_b\|_{T_\phi} \right) + \mathfrak{h}_\sigma^2 \|F_{ab}\|_{T_\phi}. \quad (5.5)$$

This is the same as what is used in [29], except we now define \mathfrak{h}_σ by (5.2).

We also need the following mass intervals. Given $\delta > 0$, let

$$\mathbb{I}_j = \begin{cases} [0, \delta) & (j < N) \\ [\delta L^{-2(N-1)}, \delta) & (j = N), \end{cases} \quad (5.6)$$

and, for $\tilde{m}^2 \in \mathbb{I}_j$, let

$$\tilde{\mathbb{I}}_j = \tilde{\mathbb{I}}_j(\tilde{m}^2) = \begin{cases} [\frac{1}{2}\tilde{m}^2, 2\tilde{m}^2] \cap \mathbb{I}_j & (\tilde{m}^2 \neq 0) \\ [0, L^{-2(j-1)}] \cap \mathbb{I}_j & (\tilde{m}^2 = 0). \end{cases} \quad (5.7)$$

Let $\tilde{s}_j = (\tilde{m}^2, \tilde{g}_j)$, and let $\tilde{\chi}_j$ be given by (4.13) with j_m determined by mass \tilde{m}^2 rather than m^2 . We extend the bulk domain of (4.21) to a domain $\mathbb{D}_j^\partial(\tilde{s}) \subset \mathcal{V}_h^{(0)} \times \mathcal{K}_j$, (with the same convention when the parameter \tilde{s} is omitted), defined by

$$\mathbb{D}_j(\tilde{s}_j) = \mathcal{D}_j \times B_{\mathcal{W}_j}(\alpha \tilde{\chi}_j \tilde{g}_j^3). \quad (5.8)$$

The domain \mathbb{D} also depends on the vector h , but we regard h as fixed and do not include it in the notation.

5.2 A single renormalisation group step including observables

The following theorem is the centrepiece of the proof of Theorems 1.1–1.3. For observables, it provides the non-perturbative counterpart to the perturbative statement of Proposition 3.2. Its proof, which requires adjustments to some arguments in [28, 29], is deferred to Section 7.

One of its consequences is that if $h \in \mathbb{R}^n$ is chosen to lie in one of the eigenspaces E^\pm , then $V_j \in \mathcal{V}_h$ for all j . In other words, the complete renormalisation group flow keeps the vector $h \in \mathbb{R}^n$ fixed for all j , it does vary during the flow. Proposition 3.2 gives the perturbative version of this fact. The norms in Theorem 5.1 depend on the choice of the eigenspace E^\pm via the appearance of $\gamma_{n,p}^\pm$ in the definition of $\mathfrak{h}_{\sigma,j}$ in (5.2), and thus the estimates it provides also depend on the choice of eigenspace for h . This is the source of the two distinct powers $\gamma_{n,p}^\pm$ for the logarithms appearing in Theorems 1.1–1.2.

Theorem 5.1. *Let $d = 4$. Let $n = 0$ and $p \geq 1$, or $n \geq 1$ and $p = 1, 2$. Let $C_{\mathcal{D}}$ and L be sufficiently large. Let $h = h^\pm \in E^\pm$, and choose $\gamma = \gamma_{n,p}^\pm$ in (4.42) and (5.2). There exist $M > 0$ and $\delta > 0$ such that for $\tilde{g} \in (0, \delta)$ and $\tilde{m}^2 \in \mathbb{I}_+$, and with the domain \mathbb{D} defined using any $\alpha > M$, the maps*

$$R_+ : \mathbb{D}(\tilde{s}) \times \tilde{\mathbb{I}}_+(\tilde{m}^2) \rightarrow \mathcal{V}_h^{(1)}, \quad K_+ : \mathbb{D}(\tilde{s}) \times \tilde{\mathbb{I}}_+(\tilde{m}^2) \rightarrow \mathcal{W}_+(\tilde{s}_+) \quad (5.9)$$

define $(U, K) \mapsto (V_+, K_+)$ as in (4.8) and obeying (4.5), and satisfy the estimates

$$\|R_+\|_{\mathcal{V}_h} \leq M \tilde{\chi}_+ \tilde{g}_+^3, \quad \|K_+\|_{\mathcal{W}_+} \leq M \tilde{\chi}_+ \tilde{g}_+^3. \quad (5.10)$$

In addition, R_+, K_+ are jointly continuous in all arguments m^2, V, K .

In particular, the bounds of (5.10) hold when $\tilde{m}^2 = m^2 \in \mathbb{I}_j$, and in this case $\tilde{\chi}_+ = \chi_{j+1}$. Also, \tilde{g}_j can be replaced in estimates by \bar{g}_j , due to (4.19). This leads to the replacement of the

right-hand sides of (5.10) by $\chi_{j+1}\bar{g}_{j+1}$, which itself can be replaced by $\chi_j\bar{g}_j$. Thus there is no need for distinction between these various options.

More can be said about R_+ , for which we have the exact formulas (4.9)–(4.10). It follows exactly as in [29, Proposition 1.14] that

$$\pi_a R_+ = \pi_b R_+ = 0 \text{ for } j \geq j_{ab}, \quad \pi_{ab} R_+ = 0 \text{ for } j < j_{ab}. \quad (5.11)$$

We write $R_+^{\lambda_x}$ for the coupling constant corresponding to λ_x in R_+ , and similarly for $R_+^{q_x}$. We write

$$f_j \prec g_j \text{ to mean that } f_j \leq c g_j. \quad (5.12)$$

By definition of the \mathcal{V}_h norm, and with (5.11), the first bound of (5.10) implies that, for $(U, K) \in \mathbb{D}_j(\bar{s}_j)$,

$$|R_+^{\lambda_x}| \prec \ell_j^{-p} \ell_{\sigma,j}^{-1} \chi_j \bar{g}_j^3 \mathbb{1}_{j < j_{ab}} \prec \Gamma_j \chi_j \bar{g}_j^2 \mathbb{1}_{j < j_{ab}}, \quad (5.13)$$

$$|R_+^{q_x}| \prec \ell_{\sigma,j}^{-2} \chi_j \bar{g}_j^3 \prec \Gamma_{j_{ab}}^2 L^{-2j_{ab}} 2^{-2p(j-j_{ab})} \chi_j \bar{g}_j \mathbb{1}_{j \geq j_{ab}}. \quad (5.14)$$

As discussed below the statement of Proposition 3.2, the first scale for which q_{pt} of (3.39) can be nonzero is $q_{\text{pt},j_{ab}+1}$. The indicator function in (5.14) shows that this remains true on a non-perturbative level.

With observables, according to [29, (1.69)], the statement for the bulk flow in (4.11) is accompanied by the statement that, for $x = a$ or $x = b$,

$$\begin{aligned} &\text{if } \pi_x V = 0 \text{ and } \pi_x K(X) = 0 \text{ for all } X \in \mathcal{P} \text{ then} \\ &\pi_x R_+ = \pi_{ab} R_+ = 0 \text{ and } \pi_x K_+(U) = \pi_{ab} K_+(U) = 0 \text{ for all } U \in \mathcal{P}_+. \end{aligned} \quad (5.15)$$

Moreover, as discussed below [29, (1.69)], $\lambda_{a,+}$ is independent of each of λ_b , $\pi_b K$, and $\pi_{ab} K$, and a similar statement holds for $\lambda_{b,+}$.

5.3 Complete renormalisation group flow

Given $(m^2, g_0) \in [0, \delta)^2$, the initial conditions for the global existence of the bulk renormalisation group flow are given by

$$\pi_\emptyset U_0 = U_0^c = (g_0, z_0^c(m^2, g_0), \nu_0^c(m^2, g_0)), \quad (5.16)$$

and this gives rise via Theorem 4.6 to the sequence $U_j(m^2, g_0)$. The next three propositions show that the flow with observables, and with initial condition $U_0 \in \mathcal{V}_h^{(0)}$ defined by

$$\pi_\emptyset U_0 = U_0^c, \quad \lambda_{x,0} \in \{0, 1\}, \quad u_0 = q_{x,0} = t_{x,0} = 0, \quad (x = a, b), \quad (5.17)$$

exists for all $j \leq N$, and they state properties of that flow. The flow of λ_x, q_x, t_x does depend on the choice of the vector $h = h^\pm \in E^\pm$, and on the choice of initial condition $\lambda_{a,0}, \lambda_{b,0}$, but we do not add labels to indicate this dependence. When $\lambda_{a,0} = 0$ or $\lambda_{b,0} = 0$, we define the coalescence scale j_{ab} to be $j_{ab} = \infty$ rather than via (3.10), since in this case at least one of the observable fields σ_a, σ_b is absent and its point a or b no longer plays a special role.

Proposition 5.2. *Let $d = 4$. Let $n = 0$ and $p \geq 1$, or $n \geq 1$ and $p = 1, 2$. Let $h = h^\pm \in E^\pm$, and choose $\gamma = \gamma_{n,p}^\pm$ in (4.42) and (5.2). Let (ζ_0, U_0) be given by (5.17), and let $K_0 = \mathbb{1}_\emptyset$. Let $N \in \mathbb{N}$ and $(m^2, g_0) \in [\delta L^{-2(N-1)}, \delta) \times (0, \delta)$. There exist (ζ_j, U_j, K_j) such that $(U_j, K_j) \in \mathbb{D}_j$ and (4.5) hold for $0 \leq j \leq N$. This choice is such that $\pi_\emptyset U_j = U_j^c(m^2, g_0)$. For $x = a$ or $x = b$, if $\lambda_{x,0} = 0$ then $\lambda_{x,j} = 0$ for all $0 \leq j \leq N$, whereas if $\lambda_{x,0} = 1$ then*

$$\lambda_{x,j} = \begin{cases} \Pi_{j-1}^\pm \left(1 + \sum_{k=0}^{j-1} r_{x,k}^\pm \right) & (j \leq j_{ab}) \\ \lambda_{x,j_{ab}-1} & (j > j_{ab}), \end{cases} \quad (5.18)$$

where $r_{x,k}^\pm \in \mathbb{R}$ obey, for some $c > 0$,

$$|r_{x,k}^\pm| \leq c \chi_k \bar{g}_k^2. \quad (5.19)$$

Also, with M given by Theorem 5.1, for all j ,

$$\|K_j\|_{\mathcal{W}_j} \leq M \chi_j \bar{g}_j^3. \quad (5.20)$$

Proof. We first observe that if $\lambda_{x,0} = 0$ then $\lambda_{x,j} = 0$ for all $0 \leq j \leq N$, due to (3.38) and (5.15). We therefore assume that $\lambda_{x,1} = 1$.

The proof is by induction on j . We make the induction hypothesis:

IH _{j} : for all $k \leq j$, $(U_k, K_k) \in \mathbb{D}_k$, (5.18) and (5.20) hold with j replaced by k ; and (5.19) holds for all $k < j$.

By direct verification, IH₀ holds since $\Pi_{-1}^\pm = 1$ and $\|K_0\|_{\mathcal{W}_0} = 0$ by definition. We now assume IH _{j} and show that it implies IH _{$j+1$} .

We apply Theorem 5.1 with $\Gamma_j = \Gamma_j^\pm$ in (5.2), where $\Gamma_j^\pm = (g_j/g_0)^{\gamma_{n,p}^\pm}$ as in (4.42). By the induction hypothesis and (5.10), $K_{j+1} \in B_{\mathcal{W}_j}(\alpha \tilde{\chi}_{j+1} \tilde{g}_{j+1})$ and satisfies (5.20). According to Theorem 4.6, the sequence U^c satisfies the bounds required for $\pi_\emptyset U$ in the definition of \mathbb{D} and obeys (4.11)–(4.12), so that $\pi_\emptyset U_j = U_j^c$ for all j . Therefore, to verify $(U_{j+1}, K_{j+1}) \in \mathbb{D}_{j+1}$, it suffices to show that $|\lambda_{x,j+1}| < C_{\mathcal{D}} \ell_{j+1}^{-p} \ell_{\sigma,j+1}^{-1}$.

Let x denote a or b . Since $R_{j+1}(U_j, K_j) \in \mathcal{V}_h$ by (5.9), the inclusion of the non-perturbative remainder $R_{j+1}^{\lambda_x}$ in the flow of λ_x gives, by (4.10) and Proposition 3.2,

$$\lambda_{x,j+1} = \begin{cases} f_j^\pm \lambda_{x,j} + R_{j+1}^{\lambda_x} & (j+1 < j_{ab}) \\ \lambda_{x,j_{ab}-1} & (j+1 \geq j_{ab}). \end{cases} \quad (5.21)$$

The flow of λ_x stops at the coalescence scale, so we restrict to $j+1 < j_{ab}$. In this case, we insert (5.18) into (5.21) to obtain

$$\lambda_{x,j+1} = \Pi_j^\pm \left(1 + \sum_{k=0}^{j-1} r_{x,k}^\pm \right) + R_{j+1}^{\lambda_x} = \Pi_j^\pm \left(1 + \sum_{k=0}^j r_{x,k}^\pm \right), \quad (5.22)$$

with $r_{x,j}^\pm = (\Pi_j^\pm)^{-1} R_{j+1}^{\lambda_x}$. By (5.13), this gives

$$|r_{x,j}^\pm| = |\Pi_j^\pm|^{-1} |R_{j+1}^{\lambda_x}| \leq c' (\Pi_j^\pm)^{-1} \Gamma_j^\pm \chi_j \bar{g}_j^2, \text{ for some } c' > 0. \quad (5.23)$$

Then (5.19) follows since $(\Pi_j^\pm)^{-1}$ and Γ_j^\pm are comparable by Lemma 4.10 and (4.42).

To complete the induction, it remains to prove that $|\lambda_{x,j+1}| \ell_{j+1}^p \ell_{\sigma,j+1} < C_{\mathcal{D}} \tilde{g}_{j+1}$. By (5.1)–(5.2), it suffices to prove that

$$|\lambda_{x,j+1}| < C_{\mathcal{D}} \Gamma_{j+1} \quad (j+1 < j_{ab}) \quad (5.24)$$

(the case of $j+1 \geq j_{ab}$ then also follows). This follows from (5.18)–(5.19), the estimate $\sum_{k=0}^j |r_{x,k}^\pm| = O(g_0)$ (by (4.14)), and Lemma 4.10, since we may assume that $C_{\mathcal{D}} > 2$. ■

Proposition 5.3. *Let $d = 4$. Let $n = 0$ and $p \geq 1$, or $n \geq 1$ and $p = 1, 2$. Let $h = h^\pm \in E^\pm$, and choose $\gamma = \gamma_{n,p}^\pm$ in (4.42) and (5.2). Let (ζ_0, U_0) be given by (5.17) with $\lambda_{a,0} = \lambda_{b,0} = 1$, and let $K_0 = \mathbb{1}_{\emptyset}$. Let $N \in \mathbb{N}$ and $(m^2, g_0) \in [\delta L^{-2(N-1)}, \delta) \times (0, \delta)$. Let a, b be such that $j_{ab} < j_m$. For $j \leq N$ and $x = a, b$, the entry $q_{x,j}$ in ζ_j produced by Proposition 5.2 obeys*

$$q_{x,j} = p! \lambda_{a,j_{ab}} \lambda_{b,j_{ab}} w_{j_{ab}}^p + \sum_{i=j_{ab}}^{j-1} R_i^{q_x}, \quad (5.25)$$

with $|R_i^{q_x}| \prec \Gamma_{j_{ab}}^2 L^{-2j_{ab}} 2^{-2p(j-j_{ab})} \chi_j \bar{g}_j \mathbb{1}_{j \geq j_{ab}}$.

Proof. By (3.39) and (4.10),

$$\delta q_{x,j+1} = \delta q_{x,\text{pt}} + R_j^{q_x} = p! \lambda_{a,j_{ab}} \lambda_{b,j_{ab}} \delta[w_{j_{ab}}^p] + R_j^{q_x}. \quad (5.26)$$

For all $j < j_{ab}$, both δq_{pt} and $R_j^{q_x}$ vanish, and summation of $\delta[w_{j_{ab}}^p]$ produces a telescoping sum, so that

$$q_{x,j} = \sum_{i=j_{ab}}^{j-1} \delta q_i = p! \lambda_{a,j_{ab}} \lambda_{b,j_{ab}} w_{j_{ab}}^p + \sum_{i=j_{ab}}^{j-1} R_i^{q_x}. \quad (5.27)$$

The desired bound on $R_j^{q_x}$ is provided by (5.14), and the proof is complete. ■

Proposition 5.4. *For $x = a, b$ and $j \leq N$, each of $\lambda_{x,j}, q_{x,j}$ is independent of N , meaning that, e.g., the finite sequence $\{\lambda_{x,1}, \dots, \lambda_{x,N}\}$ takes the same values on the torus Λ_N as on a larger tori $\Lambda_{N'}$ with $N' > N$. Also, each of $\lambda_{x,j}, q_{x,j}$ is defined as a continuous function of $(m^2, g_0) \in [0, \delta)^2$, and, for $j < j_{ab}$, $\lambda_{a,j}$ is independent of $\lambda_{b,0}$, and $\lambda_{b,j}$ is independent of $\lambda_{a,0}$.*

Proof. The proof is identical to the proof of [11, Proposition 4.3(ii)], which provides the $(n, p) = (0, 1)$ version of the statement and extends without modification to our more general context here. Note that by definition of V_+ in (4.9), $\lambda_{x,N}$ and $q_{x,N}$ are constructed from K_{N-1} and I_{N-1} , so they are independent of whether the torus has scale N , or a larger scale. ■

5.4 Inductive limit of observable flow

Proposition 5.4 permits the observable coupling constants to be defined as *infinite* sequences, not stopped at $j = N$, via an inductive limit $N \rightarrow \infty$. Indeed, since $\lambda_{x,j}, q_{x,j}$ are independent of $N > j$, we obtain sequences defined for any given $j \in \mathbb{N}_0$ by choosing any $N > j$. For the case of

initial condition $\lambda_{b,0} = 0$, we write $\lambda_{a,j}^*$ for the inductive limit of the sequence $\lambda_{a,j}$, and define $\lambda_{b,j}^*$ similarly. By (5.18),

$$\lambda_{x,j}^* = \Pi_{j-1}^\pm \left(1 + \sum_{k=0}^{j-1} r_{x,k}^\pm \right) \quad \text{for } x = a, b \text{ and } j \in \mathbb{N}_0. \quad (5.28)$$

By Proposition 5.4 and by definition,

$$\lambda_{x,j} = \lambda_{x,j \wedge (j_{ab}-1)}^* \quad (5.29)$$

for any choice of initial conditions $\lambda_{a,0}, \lambda_{b,0} \in \{0, 1\}$. The following two lemmas analyse the sequences defined by inductive limits. The constants v_x^\pm in the first lemma ostensibly depend on x , but they are shown below in Proposition 6.3 to be independent of $x = a, b$. The function $g_\infty(m^2)$ in its statement is given by Lemma 4.7.

Lemma 5.5. *Fix $h \in E^\pm$ and make the corresponding choice of $\gamma = \gamma^\pm$. Let $(m^2, g_0) \in (0, \delta)^2$. For $x = a, b$, there exist constants $v_x^\pm = 1 + O(g_0)$, such that for all $j \in \mathbb{N}_0$,*

$$\lambda_{x,j}^* = v_x^\pm \Gamma_j^\pm (1 + O(\chi_j \bar{g}_j)). \quad (5.30)$$

The limit $\lambda_{x,\infty}^*(m^2) = \lim_{j \rightarrow \infty} \lambda_{x,j}^*$ exists, and

$$\lambda_{x,\infty}^*(m^2) = v_x^\pm \left(\frac{g_\infty(m^2)}{g_0} \right)^{\gamma^\pm}. \quad (5.31)$$

On the other hand, if $\lambda_{a,0} = \lambda_{b,0} = 1$ and if $j_m > j_{ab}$, then, as $|a - b| \rightarrow \infty$ with $N > j_m > j_{ab}$,

$$\lambda_{x,j_{ab}} = v_x^\pm \left(\frac{1}{\mathfrak{b}g_0 \log |a - b|} \right)^\gamma (1 + \mathcal{E}_{ab}^{(p)}). \quad (5.32)$$

Proof. Let $r_x^\pm = \sum_{k=0}^\infty r_{x,k}^\pm$. By (5.19) and (4.14), the sum converges and is $O(g_0)$, and in addition $r_x^\pm - \sum_{k=0}^{j-1} r_{x,k}^\pm = O(\chi_j \bar{g}_j)$. Let $u_x^\pm = 1 + r_x^\pm$. Then, by (5.28),

$$\lambda_{x,j}^* = \Pi_{j-1}^\pm \left(1 + r_x^\pm + O(\chi_j \bar{g}_j) \right) = u_x^\pm \Pi_{j-1}^\pm (1 + O(\chi_j \bar{g}_j)). \quad (5.33)$$

With (4.42) and Lemma 4.10, this implies that there exists $\alpha_0 = 1 + O(g_0)$ such that

$$\lambda_{x,j}^* = \alpha_0 u_x^\pm \Gamma_j^\pm (1 + O(\chi_j \bar{g}_j)). \quad (5.34)$$

This proves (5.30) with $v_x^\pm = \alpha_0 u_x^\pm$, and then (5.31) follows immediately from the definition (4.42), Lemma 4.7, and (4.14).

The proof of (5.32) follows similarly, using (5.18) and Lemma 4.11, with Lemma 4.8 (and (4.20)) to bound the error term $O(\chi_{j_{ab}} \bar{g}_{j_{ab}})$. \blacksquare

For $m^2 \geq 0$, we write

$$G_{ab}(m^2) = (-\Delta_{\mathbb{Z}^4} + m^2)_{ab}^{-1}, \quad G_{ab} = G_{ab}(0). \quad (5.35)$$

Lemma 5.6. Fix $h \in E^\pm$, $|h| = 1$, and make the corresponding choice of $\gamma = \gamma^\pm$. Let $(m^2, g_0) \in [0, \delta)^2$. For both $x = a, b$, the limit $q_{x,\infty}(m^2, g_0) = \lim_{j \rightarrow \infty} q_{x,j}(m^2, g_0)$ exists, is continuous, and for a, b with $j_{ab} < j_m$, obeys

$$q_{x,\infty}(m^2) = p! \lambda_{a,j_{ab}} \lambda_{b,j_{ab}} G_{ab}^p(m^2) + \frac{O(\bar{g}_{j_{ab}})}{(g_0 \log |a - b|)^{2\gamma^\pm}} G_{ab}^p. \quad (5.36)$$

As $|a - b| \rightarrow \infty$,

$$q_{x,\infty}(0) = p! v_a^\pm v_b^\pm \left(\frac{1}{bg_0 \log |a - b|} \right)^{2\gamma^\pm} G_{ab}^p (1 + \mathcal{E}_{ab}^{(p)}). \quad (5.37)$$

Proof. By Proposition 5.3 and the fact that $\lim_{j \rightarrow \infty} w_{j;ab} = G_{ab}(m^2)$ by definition,

$$\lim_{j \rightarrow \infty} q_{x,j} = p! \lambda_{a,j_{ab}} \lambda_{b,j_{ab}} G_{ab}^p(m^2) + \sum_{i=j_{ab}}^{\infty} R_i^{q_x}. \quad (5.38)$$

The sum on the right-hand side converges uniformly in (m^2, g_0) by Proposition 5.3 and is therefore continuous by Proposition 5.4. By Proposition 5.3 and the fact that $\chi_j \bar{g}_j \leq O(\bar{g}_{j_{ab}})$ (see [14, Lemma 2.1(i)]), we obtain

$$\sum_{i=j_{ab}}^{\infty} R_i^{q_x} = \Gamma_{j_{ab}}^2 L^{-2j_{ab}} \sum_{i=j_{ab}}^{\infty} 2^{-2p(j-j_{ab})} \chi_j \bar{g}_j = \Gamma_{j_{ab}}^2 L^{-2j_{ab}} O(\bar{g}_{j_{ab}}). \quad (5.39)$$

Note that the exponential factor in the second sum is needed for convergence, which is not otherwise guaranteed by (4.14). Then (5.36) follows from (4.43) and (3.11), together with the fact that $G_{ab} = \frac{1}{4\pi^2} |a - b|^{-2} (1 + O(|a - b|^{-2}))$ by (1.6). Finally, (5.37) follows from Lemma 5.5. \blacksquare

6 Analysis of renormalisation group flow

We now complete the proofs of our main results Theorems 1.1–1.3. As a first step, in Section 6.1, we rewrite the correlation functions of interest in terms of derivatives of the partition function Z_N of (3.9). This rewrite permits us to prove, in Proposition 6.3, that the constants v_a^\pm, v_b^\pm in (5.37) are actually independent of a, b , and hence the asymptotic behaviour as $|a - b| \rightarrow \infty$ is given by the logarithmic and G_{ab} factors in (5.37). The derivatives of Z_N naturally lead us to study derivatives of W_N and K_N , and estimates for these are given in Section 6.2. In Section 6.3, we identify the correlation functions of Theorem 1.1 in terms of the limiting values $\lambda_{x,\infty}^*$ of Lemma 5.5 and prove Theorem 1.1. Finally, in Section 6.4, we prove Theorems 1.2–1.3.

6.1 Correlation functions and the partition function

Recall the definition of the partition function Z_N from (3.9). For $n \geq 1$, we write $Z_N(\varphi)$ to emphasise its dependence on the field φ . For $n = 0$, we write $Z_N^0(\phi, \bar{\phi})$ for the degree-zero part of the form Z_N . For $n \geq 1$, we define the (un-normalised) *pressure*

$$P_N(\varphi) = \log Z_N(\varphi). \quad (6.1)$$

We use the notation used in Section 2.5 for derivatives with respect to external and observable fields. We also write $D_{\bar{\phi}}Z_N^0$ for the directional derivative of Z_N with respect to $\bar{\phi}$ in the direction of the constant field 1, evaluated at $\phi = \bar{\phi} = 0$.

Lemma 6.1. *Fix $m^2 > 0$ and $z_0 > -1$. For $n \geq 1$ and $p = 1, 2$,*

$$\langle \varphi_a^p \cdot h \rangle_{g, \nu, N} = (1 + z_0)^{p/2} D_{\sigma_a} P_N(0), \quad (6.2)$$

$$\langle \varphi_a^p \cdot h; \varphi_b^p \cdot h \rangle_{g, \nu, N} = (1 + z_0)^p D_{\sigma_a, \sigma_b}^2 P_N(0), \quad (6.3)$$

$$\langle (\varphi, H)^p; \varphi_a^p \cdot h \rangle_{g, \nu, N} = \frac{(1 + z_0)^p}{m^{2p}} D_{\varphi}^p(H) D_{\sigma_a} P_N. \quad (6.4)$$

For $n = 0$ and $p \geq 1$,

$$W_{ab, N}^{(p)}(g, \nu) = (1 + z_0)^p D_{\sigma_a \sigma_b}^2 Z_N^0(0), \quad (6.5)$$

$$S_N^{(p)}(g, \nu) = \frac{(1 + z_0)^p}{m^{2p}} D_{\bar{\phi}}^p D_{\sigma_a} Z_N^0. \quad (6.6)$$

Proof. We first prove (6.2)–(6.4). The identity (6.3) is the same as (2.27), and (6.2) also follows similarly from explicit differentiation. For (6.4), we write $\Sigma(J) = \mathbb{E}_C e^{-V_0(\Lambda) + (\varphi, J)}$, and (2.35) becomes

$$\langle (\varphi, H)^p; \varphi_a^p \cdot h \rangle_{g, \nu, N} = (1 + z_0)^p D_J^p(H) D_{\sigma_a} \log \Sigma(J). \quad (6.7)$$

As in [15, (4.9)], we obtain

$$\Sigma(J) = \mathbb{E}_C (e^{-V_0(\Lambda) + (J, \varphi)}) = e^{\frac{1}{2}(J, CJ)} Z_N(CJ) \quad (6.8)$$

by the translation $\varphi \mapsto \varphi + CJ$ to complete the square in the middle member of (6.8). This gives $\log \Sigma(J) = \frac{1}{2}(J, CJ) + \log Z_N(CJ)$. Since (J, CJ) is independent of the observable field σ_a ,

$$D_J^p(H) D_{\sigma_a} \log \Sigma(J) = D_J^p(H) D_{\sigma_a} \log Z_N(CJ). \quad (6.9)$$

Since H is a constant field, $CH = m^{-2}H$. The chain rule then gives

$$D_J^p(H) D_{\sigma_a} \log \Sigma(J) = m^{-2p} D_{\varphi}^p(H) D_{\sigma_a} \log Z_N, \quad (6.10)$$

and the proof of (6.4) is complete.

The case $n = 0$ is similar, except the logarithm is superfluous due to the self-normalisation property of the Gaussian super-expectation. The identity (6.5) is a restatement of (2.33). For (6.6), we define $\Sigma(J, \bar{J}) = \mathbb{E}_C e^{-V_0(\Lambda) + (J, \bar{\phi}) + (\bar{J}, \phi)}$, and rewrite (2.36) as

$$S_N^{(p)}(g, \nu) = (1 + z_0)^p D_J^p D_{\sigma_a} \Sigma(J, \bar{J}). \quad (6.11)$$

Now completion of the square (as in [12, (4.23)]) gives

$$\Sigma(J, \bar{J}) = e^{(J, C\bar{J})} Z_N^0(CJ, C\bar{J}), \quad (6.12)$$

and (6.6) again follows by differentiation and the chain rule. ■

6.2 Non-perturbative estimates

The following lemma permits the control of non-perturbative quantities in the proofs of our main theorems. To simplify its notation, we write

$$f_{ab} = \frac{1}{2^{p(N-j_{ab})+}} \frac{1}{|a-b|^p} \frac{1}{(g \log |a-b|)^\gamma}. \quad (6.13)$$

We write D_σ^k to mean no derivative for $k = 0$, the derivative with respect to σ_a for $k = 0, 1, 2$, and derivatives with respect to σ_a and σ_b for $k = 2$.

Lemma 6.2. *Let $h = h^\pm \in E^\pm$, and let $\gamma = \gamma^\pm$. For $n = 0$, $p \geq 1$, the following estimates (all at zero field) hold uniformly in $g \in (0, \delta)$ and $m^2 \in [\delta L^{-2(N-1)}, \delta)$. For initial conditions $\lambda_{a,0} = \lambda_{b,0} = 1$ and for $l = 0, 1, 2$,*

$$|D_\sigma^l K_N^0(\Lambda)| \prec \chi_N \bar{g}_N^{3-l} f_{ab}^l. \quad (6.14)$$

For initial conditions $\lambda_{a,0} = 1$, $\lambda_{b,0} = 0$, and for $k = 0, 1, \dots, p$ and $l = 0, 1$,

$$|D_\phi^k D_{\sigma_a}^l K_N^0(\Lambda)| \prec \chi_N \bar{g}_N^{3-l} \frac{L^{N(k-lp)}}{(g_0 \log m^{-2})^{l\gamma}}, \quad (6.15)$$

$$|D_\phi^k D_{\sigma_a}^l W_N^0(\Lambda)| \prec \chi_N \bar{g}_N^{2-l} \frac{L^{N(k-lp)}}{(g_0 \log m^{-2})^{l\gamma}}. \quad (6.16)$$

The bounds (6.14)–(6.16) also hold for $n \geq 1$ and $p = 1, 2$, after changing K_N^0 to K_N and making directional derivatives with respect to φ in the direction of a constant field 1.

Proof. We give the proof for $n = 0$. The proof for $n \geq 1$ involves only slight changes in notation. By (4.19), \tilde{g}_j and \bar{g}_j are interchangeable in estimates.

Recall the definitions of the $T_{0,j}(\ell_j)$ and $\Phi_j(\ell_j)$ norms from [12, Section 6.3]. In (5.5), in the $T_{0,j}(\ell_j)$ norm each occurrence of σ or $\bar{\sigma}$ produces the weight

$$\ell_{\sigma,j} = \ell_0^{-p} \Gamma_{j \wedge j_{ab}}^{-1} 2^{p(j-j_{ab})+} L^{p(j \wedge j_{ab})} \tilde{g}_j \quad (6.17)$$

defined in (5.2). We apply [29, (1.62)] which uses this fact, together with (5.20), to see that for $l = 0, 1, 2$ the bound

$$\begin{aligned} |D_\sigma^l K_N^0(\Lambda; 0, 0)| &\leq \ell_{\sigma,j}^{-l} \|K_N(\Lambda)\|_{T_{0,N}(\ell_N)} \leq \ell_{\sigma,j}^{-l} \|K_N\|_{\mathcal{W}_N} \\ &\prec \Gamma_{j_{ab}}^l 2^{-lp(N-j_{ab})+} L^{-lpj_{ab}} \chi_N \bar{g}_N^{3-l} \end{aligned} \quad (6.18)$$

holds uniformly in $m^2 \in [\delta L^{-2(N-1)}, \delta)$. By (3.11), $L^{-j_{ab}} \prec |a-b|^{-1}$. The logarithmic behaviour of $\Gamma_{j_{ab}}$ is given by (4.43), and (6.14) is proved.

For any $k \leq p_N$, $l = 0, 1$, $F \in \mathcal{N}$, and test functions $J_i : \Lambda \rightarrow \mathbb{C}$ ($i = 1, \dots, k$), it follows from the definition of the $T_{0,N}(\ell_N)$ norm that

$$|D_\phi^k D_{\sigma_a}^l F^0(0, 0; J_1, \dots, J_p)| \leq \ell_{\sigma,N}^{-l} \|F\|_{T_{0,N}(\ell_N)} \|J_1\|_{\Phi_N(\ell_N)} \cdots \|J_k\|_{\Phi_N(\ell_N)}. \quad (6.19)$$

By definition, $\|1\|_{\Phi_N(\ell_N)} = \ell_N^{-1}$ (as in [12, (8.55)]). As in (6.18), this gives

$$|D_\phi^k D_{\sigma_a}^l K_N^0(\Lambda; 0, 0; 1, \dots, 1)| \leq \ell_{\sigma,N}^{-l} \|K_N(\Lambda)\|_{T_{0,N}(\ell_N)} \|1\|_{\Phi_N(\ell_N)}^k \leq \ell_{\sigma,N}^{-l} \ell_N^{-k} \|K_N\|_{\mathcal{W}_N}. \quad (6.20)$$

With the initial conditions assumed for (6.15)–(6.16), we have $j_{ab} = \infty$. By (6.17), (5.1), (4.42), (4.25), and (4.23),

$$\ell_{\sigma_a N}^{-l} \ell_N^{-k} = \tilde{g}_N^{-l} \Gamma_N^l (\ell_0 L^{-N})^{lp-k} \prec \bar{g}_N^{-l} (g_0 \log m^{-2})^{-l\gamma} L^{N(k-lp)}. \quad (6.21)$$

With (5.20), this proves (6.15). Finally, for the bound on W_N , we recall from [28, Proposition 4.1] that

$$\|W_N(\Lambda)\|_{T_{0,N}} \prec \chi_N \bar{g}_N^2, \quad (6.22)$$

and (6.16) then follows exactly as in (6.20). \blacksquare

6.3 Proof of Theorem 1.1

Let $n \geq 0$. For small $g, \varepsilon > 0$, set $\nu = \nu_c + \varepsilon$, and let $(m^2, g_0, \nu_0, z_0) = (\tilde{m}^2, \tilde{g}_0, \tilde{\nu}_0, \tilde{z}_0)$ be the functions of (g, ε) given by Proposition 4.12. This choice is consistent with the initial condition (5.17) that guarantees the existence of the global flow with observables. By [12, (4.34)] for $n = 0$, and [15, (4.24)] for $n \geq 1$, it provides the *identity*

$$\chi = \chi(g, \nu) = \frac{1 + \tilde{z}_0}{\tilde{m}^2} = \frac{1 + z_0}{m^2}. \quad (6.23)$$

Proposition 6.3. *Let $h = h^\pm \in E^\pm$ and $\gamma = \gamma^\pm$. Let $n = 0$ and $p \geq 1$, or $n \geq 1$ and $p = 1, 2$. For $n \geq 1$, let H be a constant field with value H_0 . For $(g, \varepsilon) \in (0, \delta)^2$, and for $x = a, b$,*

$$\frac{1}{\chi^p} S^{(p)}(g, \nu) = p! \lambda_{x, \infty}^* \quad (n = 0) \quad (6.24)$$

$$\frac{1}{\chi^p} \langle (\varphi, H)^p; \varphi_a^p \cdot h \rangle_{g, \nu} = p! (H_0^p \cdot h) \lambda_{x, \infty}^* \quad (n \geq 1). \quad (6.25)$$

In particular, the infinite volume limit on the left-hand side of (6.25) exists.

Proof. We use initial conditions $\lambda_{a,0} = 1$ and $\lambda_{b,0} = 0$. We start with (4.6), but without setting the fields to zero, to get

$$Z_N = e^{\zeta_N} (I_N + K_N), \quad (6.26)$$

where Z_N, I_N, K_N depend on $(\phi, \bar{\phi})$ for $n = 0$, and on φ for $n \geq 1$.

We first prove (6.24). In this case, $\zeta_N = \frac{1}{2}(q_{a,N} + q_{b,N})\sigma_a\sigma_b$. By (3.26) (since Λ is a single block at scale N),

$$Z_N^0 = e^{\zeta_N} (I_N^0 + K_N^0) = e^{\zeta_N} (e^{-V_N^0} (1 + W_N^0) + K_N^0). \quad (6.27)$$

Since $\pi_\emptyset e^{-V_N^0} = e^{-U_N^0}$ and $D_{\sigma_a} e^{-V_N^0} = \lambda_{a,N}^* \bar{\phi}_a^p$,

$$D_{\sigma_a} Z_N^0 = \lambda_{a,N}^* \bar{\phi}_a^p e^{-U_N^0} (1 + \pi_\emptyset W_N^0) + e^{-U_N^0} D_{\sigma_a} W_N^0 + D_{\sigma_a} K_N^0. \quad (6.28)$$

We differentiate with respect to $\bar{\phi}$ in direction 1, p times, and set $\phi = \bar{\phi} = 0$ to obtain

$$D_{\bar{\phi}}^p D_{\sigma_a} Z_N^0 = p! \lambda_{a,N}^* + D_{\bar{\phi}}^p D_{\sigma_a} W_N^0 + D_{\bar{\phi}}^p D_{\sigma_a} K_N^0, \quad (6.29)$$

where we used the facts that $e^{-U_N^0} = 1$ and $W_N^{0,\emptyset} = 0$ when $\phi = \bar{\phi} = 0$. By Lemma 6.2 and (6.6),

$$\left(\frac{1+z_0}{m^2}\right)^{-p} S_N^{(p)}(g, \nu) = D_\phi^p D_{\sigma_a} Z_N^0 = p! \lambda_{a,N}^* + O\left(\frac{\chi_N \bar{g}_N}{(g \log m^{-2})^{\gamma^+}}\right) \quad (6.30)$$

(since $\gamma^- = \gamma^+$ for $n = 0$). We let $N \rightarrow \infty$ in (6.30), using the facts that $\chi_N \bar{g}_N \rightarrow 0$ by (4.14), $S_N^{(p)} \rightarrow S^{(p)}$ by Proposition 2.1, and $\lambda_{a,N}^* \rightarrow \lambda_{a,\infty}^*$ by Lemma 5.5. We also use (6.23) to identify the factor χ^{-p} on the left-hand side. This proves (6.24).

To prove (6.25), we apply both $D_\phi^p D_{\sigma_a}$ to the logarithm of the right-hand side of (6.26). Since ζ_N is independent of φ , in terms of the pressure (6.1) this gives

$$D_\phi^p D_{\sigma_a} P_N = D_\phi^p D_{\sigma_a} \log(I_N + K_N). \quad (6.31)$$

By definition, $I_N = e^{-U_N}(1 + W_N)$, and we write $I_N + K_N = e^{-U_N}(1 + W_N + e^{U_N} K_N)$. Since $D_\phi^p D_{\sigma_a}(-U_N) = p! \lambda_{a,N}^*(H_0^p \cdot h)$,

$$D_\phi^p D_{\sigma_a} P_N = p! \lambda_{a,N}^*(H_0^p \cdot h) + D_\phi^p D_{\sigma_a} \log(1 + W_N + e^{U_N} K_N). \quad (6.32)$$

It is now an exercise in calculus to apply Lemma 6.2 (and the fact that U_N lies in the domain \mathcal{D}_N of (5.4)) to conclude that

$$D_\phi^p D_{\sigma_a} \log(1 + W_N + e^{U_N} K_N) = O\left(\frac{\chi_N \bar{g}_N}{(g \log m^{-2})^\gamma}\right). \quad (6.33)$$

Then, by (6.4),

$$\left(\frac{1+z_0}{m^2}\right)^{-p} \langle (\varphi, H)^p; \varphi_a^p \cdot h \rangle_{g,\nu,N} = p! \lambda_{a,N}^*(H_0^p \cdot h) + O\left(\frac{\chi_N \bar{g}_N}{(g \log m^{-2})^\gamma}\right). \quad (6.34)$$

Again we use $\chi_N \bar{g}_N \rightarrow 0$ and $\lambda_{a,N}^* \rightarrow \lambda_{a,\infty}^*$ to see that the limit as $N \rightarrow \infty$ of the right-hand side exists and equals the right-hand side of (6.25). Therefore, the limit of the left-hand side must also exist and so

$$\langle (\varphi, H)^p; \varphi_a^p \cdot h \rangle_{g,\nu} = \lim_{N \rightarrow \infty} \langle (\varphi, H)^p; \varphi_a^p \cdot h \rangle_{g,\nu,N} \quad (6.35)$$

exists in the sense of (1.9). We complete the proof of (6.25) by appealing to (6.23). \blacksquare

Corollary 6.4. *The constant v_x^\pm in (5.30) is, in fact, independent of x , and moreover, for $p = 1$ and for all $n \geq 0$, $\lambda_{x,\infty}^* = 1$.*

Proof. Since the left-hand sides of (6.24) and (6.25) are independent of x , $\lambda_{x,\infty}^*$ and v_x^\pm must also be independent of x .

Let $p = 1$. By definition, $S^{(1)}$ is the susceptibility χ , so (6.24) yields $\lambda_{x,\infty}^* = 1$ (as was proved in [11, Lemma 4.6]). For $n \geq 1$, since $(\varphi, H) = \sum_x \varphi_x \cdot H_0$, we take $H_0 = \hat{e}_1$, the first standard basis vector. Using $\varphi_x^i \mapsto -\varphi_x^i$ symmetry and (1.3),

$$\langle (\varphi, H); \varphi_a \cdot h \rangle_{g,\nu} = \lim_{N \rightarrow \infty} \sum_{x \in \Lambda_N} \langle \varphi_x^1; \varphi_a \cdot h \rangle_{g,\nu,N} = \lim_{N \rightarrow \infty} \sum_{x \in \Lambda_N} h^1 \langle \varphi_x^1 \varphi_a^1 \rangle_{g,\nu,N} = (H_0 \cdot h) \chi. \quad (6.36)$$

Thus (6.25) simplifies to $\lambda_{x,\infty}^* = 1$. \blacksquare

Proof of Theorem 1.1. (i) By (1.4) and (1.14), (6.23) gives

$$m^2 \sim (1 + z_0)A_g^{-1}\varepsilon(\log \varepsilon^{-1})^{-\gamma^+} \quad \text{as } \varepsilon \downarrow 0, \quad (6.37)$$

and hence $\log m^{-2} \sim \log \varepsilon^{-1}$. Using (5.31) and Lemma 4.7, and since $g_0 = g(1 + O(g))$,

$$\lambda_\infty^*(m^2) \sim \frac{\tilde{v}^\pm}{(\log \varepsilon^{-1})^{\gamma^\pm}}, \quad \tilde{v}^\pm = \frac{v^\pm}{(g_0 \mathbf{b})^{\gamma^\pm}} = \frac{1}{(g \mathbf{b})^{\gamma^\pm}}(1 + O(g)), \quad (6.38)$$

where, in view of Corollary 6.4, we have dropped the labels x on λ_∞^* and v^\pm . For $n = 0$, we use (6.24) (recall that $\gamma^+ = \gamma^-$ for $n = 0$) to obtain

$$\frac{1}{\chi^p} S^{(p)}(g, \nu) = p! \lambda_\infty^*(m^2) \sim p! \frac{\tilde{v}^+}{(\log \varepsilon^{-1})^{\gamma^+}}. \quad (6.39)$$

This proves (1.20).

(ii) Let $n \geq 1$ and $p = 2$. Now we use (6.25) to obtain

$$\frac{1}{\chi^2} \langle (\varphi, H)^2; \varphi_a^2 \cdot h \rangle_{g, \nu} = 2! (H_0^2 \cdot h) \lambda_{x, \infty}^* \sim (H_0^2 \cdot h) \frac{2\tilde{v}^\pm}{(\log \varepsilon^{-1})^{\gamma^\pm}}, \quad (6.40)$$

where H_0 is the constant value of the field H , and $H_0^2 \in \mathbb{R}^n$ is the vector whose components are the squares of the components of H_0 . For the choice $h = n^{-1/2} e^+ \in E^+$, we have $\varphi_a^2 \cdot h = n^{-1/2} |\varphi_a|^2$ and $H_0^2 \cdot h = n^{-1/2} |H_0|^2$. We cancel the $n^{-1/2}$ factor on both sides of (6.40) and obtain

$$\frac{1}{\chi^2} \langle (\varphi, H)^2; \varphi_a^2 \cdot h \rangle_{g, \nu} \sim |H_0|^2 \frac{2\tilde{v}^+}{(\log \varepsilon^{-1})^{\gamma^+}}. \quad (6.41)$$

We take $H_0 = \hat{e}_k$ to be the k^{th} standard basis vector, and then sum over k , to obtain

$$\frac{1}{\chi^2} \sum_{x, y} \langle \varphi_x \cdot \varphi_y; |\varphi_a|^2 \rangle_{g, \nu} = \frac{1}{\chi^2} \sum_{x, y} \sum_{k=1}^n \langle \varphi_x^k \varphi_y^k; |\varphi_a|^2 \rangle_{g, \nu} \sim \frac{2n\tilde{v}^+}{(\log \varepsilon^{-1})^{\gamma^+}}. \quad (6.42)$$

This proves (1.21). Suppose now that $n \geq 2$. By symmetry, (6.42) gives

$$\frac{1}{\chi^2} \sum_{x, y} \langle \varphi_x^1 \varphi_y^1; (\varphi_a^1)^2 \rangle_{g, \nu} + (n-1) \frac{1}{\chi^2} \sum_{x, y} \langle \varphi_x^1 \varphi_y^1; (\varphi_a^2)^2 \rangle_{g, \nu} \sim \frac{2\tilde{v}^+}{(\log \varepsilon^{-1})^{\gamma^+}}. \quad (6.43)$$

In (6.40) we take $H_0 = \hat{e}_1$ and $h = 2^{-1/2}(1, -1, 0, \dots, 0) \in E^-$. Since $h \in E^-$, now $\gamma = \gamma^-$. We obtain

$$\frac{1}{\chi^2} \sum_{x, y} \langle \varphi_x^1 \varphi_y^1; (\varphi_a^1)^2 \rangle_{g, \nu} - \frac{1}{\chi^2} \sum_{x, y} \langle \varphi_x^1 \varphi_y^1; (\varphi_a^2)^2 \rangle_{g, \nu} \sim \frac{2\tilde{v}^-}{(\log \varepsilon^{-1})^{\gamma^-}}. \quad (6.44)$$

Since $\gamma^- < \gamma^+$, the combination of (6.43)–(6.44) gives

$$\frac{1}{\chi^2} \sum_{x, y} \langle \varphi_x^1 \varphi_y^1; (\varphi_a^1)^2 \rangle_{g, \nu} \sim \frac{n-1}{n} \frac{2\tilde{v}^-}{(\log \varepsilon^{-1})^{\gamma^-}}, \quad (6.45)$$

$$\frac{1}{\chi^2} \sum_{x, y} \langle \varphi_x^1 \varphi_y^1; (\varphi_a^2)^2 \rangle_{g, \nu} \sim -\frac{1}{n} \frac{2\tilde{v}^-}{(\log \varepsilon^{-1})^{\gamma^-}}, \quad (6.46)$$

which proves (1.22)–(1.23).

(iii) The asymptotic formula (1.24) follows from (6.38), and the proof is complete. \blacksquare

6.4 Proof of Theorems 1.2–1.3

Proof of Theorem 1.2. (i-ii) We denote the parameters (n, p) by superscripts. The infinite volume limit of the watermelon network can be computed as a limit using Proposition 2.1, and for $n \geq 1$ we have defined the critical infinite volume limits of correlation functions as in (1.8). For $n \geq 1$, let $(S_c)_{ij} = \langle (\varphi_a^i)^p; (\varphi_b^j)^p \rangle_{\nu_c(n)}$ denote the matrix of critical correlation functions. According to (6.3) and (6.5) (we drop the notation for evaluation at zero as all fields are evaluated at zero here),

$$(1 + \tilde{z}_0(g, 0))^p \lim_{\varepsilon \downarrow 0} \lim_{N \rightarrow \infty} D_{\sigma_a \sigma_b}^2 P_N^{(n,p)} = \begin{cases} W_{ab}^{(p)}(\nu_c(0)) & (n = 0, p \geq 1), \\ h \cdot S_c h & (n \geq 1, p = 1, 2). \end{cases} \quad (6.47)$$

For the prefactor on the left-hand side, we have used Proposition 4.12 for existence of the limit $\tilde{z}_0(g, \varepsilon) \rightarrow \tilde{z}_0(g, 0)$ as $\varepsilon \downarrow 0$. It also follows from Theorem 4.6 that $\tilde{z}_0 = O(g)$.

By (4.6),

$$P_N^{(n,p)} = \log Z_N^{(n,p)} = \zeta_N + \log(1 + K_N(\Lambda)), \quad (6.48)$$

with $\zeta_N = \frac{1}{2}(q_{a,N} + q_{b,N})\sigma_a\sigma_b + t_{a,N}\sigma_a + t_{b,N}\sigma_b + u_N|\Lambda|$ (if $n = 0$ then $t_{a,N} = t_{b,N} = u_N = 0$). Differentiation gives

$$D_{\sigma_a \sigma_b}^2 P_N^{(n,p)} = \frac{1}{2}(q_{a,N} + q_{b,N}) + \frac{D_{\sigma_a \sigma_b}^2 K_N}{1 + \pi_\emptyset K_N} - \frac{(D_{\sigma_a} K_N)(D_{\sigma_b} K_N)}{(1 + \pi_\emptyset K_N)^2}. \quad (6.49)$$

According to (6.14), the last two terms vanish in the $N \rightarrow \infty$ limit. We write v^\pm for the common value of v_a^\pm and v_b^\pm (recall Corollary 6.4). By Lemma 5.6,

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \lim_{N \rightarrow \infty} D_{\sigma_a \sigma_b}^2 P_N^{(n,p)} &= \frac{1}{2}(q_{a,\infty}(0) + q_{b,\infty}(0)) \\ &= p!(v^\pm)^2 \left(\frac{1}{\mathfrak{b}g_0 \log|a-b|} \right)^{2\gamma^\pm} G_{ab}^p \left(1 + \mathcal{E}_{ab}^{(p)} \right). \end{aligned} \quad (6.50)$$

When we carry out the renormalisation group analysis with the choice $h = h^+ = n^{-1/2}e^+ \in E^+$, it is the exponent γ^+ that occurs in (6.50), and we conclude (1.27)–(1.29) (for (1.28) we use $\gamma^+ = 0$ when $p = 1$).

(iii) Next we prove (1.30)–(1.31). Let $n \geq 2$ and $p = 2$. We make the two choices $h^+ = n^{-1/2}e^+ \in E^+$ and $h^- = 2^{-1/2}(1, -1, 0, \dots, 0) \in E^-$, which obey $|h^\pm| = 1$. By symmetry,

$$\langle \varphi_a^2 \cdot h^+; \varphi_b^2 \cdot h^+ \rangle_{\nu_c} = \langle (\varphi_a^1)^2; (\varphi_b^1)^2 \rangle_{\nu_c} + (n-1) \langle (\varphi_a^1)^2; (\varphi_b^2)^2 \rangle_{\nu_c}, \quad (6.51)$$

$$\langle \varphi_a^2 \cdot h^-; \varphi_b^2 \cdot h^- \rangle_{\nu_c} = \langle (\varphi_a^1)^2; (\varphi_b^1)^2 \rangle_{\nu_c} - \langle (\varphi_a^1)^2; (\varphi_b^2)^2 \rangle_{\nu_c}, \quad (6.52)$$

and hence

$$n \langle (\varphi_a^1)^2; (\varphi_b^1)^2 \rangle_{\nu_c} = \langle \varphi_a^2 \cdot h^+; \varphi_b^2 \cdot h^+ \rangle_{\nu_c} + (n-1) \langle \varphi_a^2 \cdot h^-; \varphi_b^2 \cdot h^- \rangle_{\nu_c}, \quad (6.53)$$

$$n \langle (\varphi_a^1)^2; (\varphi_b^2)^2 \rangle_{\nu_c} = \langle \varphi_a^2 \cdot h^+; \varphi_b^2 \cdot h^+ \rangle_{\nu_c} - \langle \varphi_a^2 \cdot h^-; \varphi_b^2 \cdot h^- \rangle_{\nu_c}. \quad (6.54)$$

The first term on the right-hand sides has been computed already in the proof of (1.29). For the second term, we instead use $h = h^-$, and now obtain (6.50) with $\gamma = \gamma^-$. This leads to (1.30)–(1.31).

(iv) The asymptotic formula (1.32) for the amplitudes $A'_{n,p,\pm}$ follows directly, using the amplitude $\frac{1}{(2\pi)^2}$ for G_{ab} in (1.6) and (6.38). \blacksquare

Proof of Theorem 1.3. We must show that

$$W_{aa}^{(p)}(\nu_c(0)) = G_{aa}^p(p! + O(g)) \quad (p \geq 1), \quad (6.55)$$

$$\langle \varphi_a^1; \varphi_a^1 \rangle_{\nu_c(n)} = G_{aa}(1 + O(g)) \quad (n \geq 1), \quad (6.56)$$

$$\langle |\varphi_a|^2; |\varphi_a|^2 \rangle_{\nu_c(n)} = G_{aa}^2(2!n + O(g)) \quad (n \geq 1), \quad (6.57)$$

$$\langle (\varphi_a^1)^2; (\varphi_a^2)^2 \rangle_{\nu_c(n)} = O(g) \quad (n \geq 2). \quad (6.58)$$

Now the coalescence scale is $j_{aa} = 0$, and hence $\lambda_j = 1$ for all j . Also, (5.2) now gives $\ell_{\sigma,j} = 2^{pj} \tilde{g}_j$, and (6.14) is replaced by

$$|D_\sigma^k K_N(\Lambda)| \prec \chi_N \bar{g}_N^{3-k} 2^{-kpN}. \quad (6.59)$$

Minor changes to the proof of Lemma 5.6 show that for the case of $a = b$ we obtain

$$\frac{1}{2}(q_{a,\infty}(0) + q_{b,\infty}(0)) = p! G_{aa}^p + O(g_0), \quad (6.60)$$

and using this in place of (6.50) leads to the desired results. In particular, the main terms cancel now in (6.54), leading to (6.58). \blacksquare

7 Proof of Theorem 5.1

Theorem 5.1 is an adaptation of [29, Theorems 1.10–1.11] to include more general observables. Its proof requires modification to some aspects of [28, 29], which focus specifically on the case of $p = 1$ and WSAW, to handle arbitrary $p \geq 1$ for WSAW, and $p = 1, 2$ for $|\varphi|^4$. These modifications can be sorted into three categories:

- (i) Different choices of parameters and changes to stability estimates are small details, which are provided in Section 7.1.
- (ii) Modification needed in one aspect of the renormalisation group map (4.8) is also a small detail, which is discussed in Section 7.2.
- (iii) For $n \geq 2$ and $h \in E^\pm$, we use new ideas to prove that the full non-perturbative flow of the coupling constants remains in the space \mathcal{V}_h . This is seen perturbatively in Proposition 3.2, and non-perturbatively from the fact that R_+ maps into \mathcal{V}_h in Theorem 5.1. The new ingredient is the requirement of h -factorisability in Definition 4.5, and the fact that this property is preserved by the renormalisation group map. We discuss this in Section 7.3.

7.1 Choices of parameters, stability and regularity estimates

7.1.1 Restriction to real coupling constants

Complex coupling constants are used in [29] only to enable Cauchy estimates in the proof of Theorems 4.6 and 5.1, but otherwise complex coupling constants are not used. In [29], real (V, K) does indeed yield real (R_+, K_+) for $|\varphi|^4$, as the vector space \mathcal{K} is a real vector space, and when V is real there is no way to produce an imaginary part in R_+ or K_+ . For WSAW, the complex field can be reexpressed in terms of a real field, and the bulk coupling constants g, ν, z, y can be seen to remain real. For the observable coupling constants λ, q , the complexity plays a more prominent role and we have not ruled out the possibility that λ, q become complex. We permit them to be complex here, and this creates no difficulties.

7.1.2 Choice of h_σ

Our choices of \tilde{g}_j and \mathfrak{h}_j in (4.18) and (5.1) are identical to those used in [28, 29], but the choice of $\mathfrak{h}_{\sigma,j}$ in (5.2) differs by the appearance of Γ_j (and thus $\gamma = \gamma_{n,p}^\pm$) and by allowing all $p \geq 1$. By [13, (6.101)], $\frac{1}{2}\tilde{g}_{j+1} \leq \tilde{g}_j \leq 2\tilde{g}_{j+1}$. Therefore, by (5.2),

$$\frac{\mathfrak{h}_{\sigma,j+1}}{\mathfrak{h}_{\sigma,j}} \leq \text{const} \begin{cases} L^p & j < j_{ab} \\ 1 & j \geq j_{ab}, \end{cases} \quad (7.1)$$

where the improved bound occurs for $j \geq j_{ab}$ since the power of L in (5.2) stops changing at the coalescence scale. On the other hand, it is indicated in [28, (1.79)] that what is required in [28, 29] is that (7.1) should hold with L^p replaced by L^1 , which is a stronger requirement than (7.1).

The L^p growth in (7.1) can be accommodated because now we take $d_+(a) = d_+(b) = p\mathbb{1}_{j < j_{ab}}$ (see Section 3.2.3), rather than the choice $\mathbb{1}_{j < j_{ab}}$ used in [28, Section 4.2.2]. Because of this, in the proof of [28, Propositions 2.8, 4.9], the computation of the small parameter $\gamma_{\alpha,\beta}(Y)$ (not to be confused with $\gamma_{n,p}^\pm$ despite its similar name) gives exactly the same value $\gamma_{\alpha,\beta}(Y) = L^{-d-1} + L^{-1}\mathbb{1}_{Y \cap \{a,b\} \neq \emptyset}$ present in [28, Proposition 2.8], and the analysis of [28, 29] can continue to be based on the crucial contraction [28, Proposition 2.8] which remains unchanged.

7.1.3 Choice of $p_{\mathcal{N}}$

By the definition of $\mathfrak{h}_{\sigma,j}$ in (5.2),

$$\frac{\ell_{\sigma,j}}{h_{\sigma,j}} = \tilde{g}_j^{1-p/4}, \quad (7.2)$$

and this grows for $p > 4$. This plays a role in [29, Lemma 2.4], which is the place that determines the choice $p_{\mathcal{N}} = 10$ used in [11, 12, 15]. We continue to use $p_{\mathcal{N}} = 10$ when $p < 4$. For $p \geq 4$ (which we consider only for WSAW), we take a larger choice, as follows.

First, [29, Lemma 2.4] is proved using [26, Proposition 3.17], which in turn relies on [29, Proposition 3.11]. We must choose $p_{\mathcal{N}} \geq A + 1$, where A appears in the proof of [29, Proposition 3.11]. In the factor $\rho^{(A+1)}$ in [29, Proposition 3.11], there can appear at most two bad ratios (7.2), since the worst case contains two observable fields, together with at least $A - 1$ good ratios ℓ_j/h_j which each yield a factor $\tilde{g}_j^{1/4}$ by (5.1). Thus, at worst, $\rho^{(A+1)}$ gives

$$\tilde{g}_j^{(A-1)/4} \tilde{g}_j^{2(1-p/4)}, \quad (7.3)$$

and we require in [29, Lemma 2.4] that this is at most $\tilde{g}_j^{10/4}$. Therefore, the minimal $p_{\mathcal{N}}$ we can permit is

$$p_{\mathcal{N}} = A + 1 \quad \text{where} \quad \frac{1}{4}(A - 1) + 2 - \frac{p}{2} \geq \frac{10}{4}, \quad \text{i.e.,} \quad A = 2p + 3. \quad (7.4)$$

Thus we can take any fixed $p_{\mathcal{N}} \geq \max\{10, 2p + 4\}$.

7.1.4 Stability estimate: value of ϵ_V

The term $(|\lambda_a| + |\lambda_b|)\mathfrak{h}_j\mathfrak{h}_{\sigma,j}$ appears in the definition of $\epsilon_{V,j}$ in [28, (1.80)], for the estimates of [28, Proposition 1.4]. This term arises as the T_0 norm of $\lambda_a\sigma_a\phi_a + \lambda_b\bar{\sigma}_b\phi_b$, and is suitable for $p = 1$. For general $p \geq 1$, it needs replacement by $(|\lambda_a| + |\lambda_b|)\mathfrak{h}_j^p\mathfrak{h}_{\sigma,j}$. This replacement has been incorporated into the definition (5.4) of \mathcal{D}_j , so that membership in \mathcal{D}_j implies that $(|\lambda_a| \vee |\lambda_b|)\ell_j^p\ell_{\sigma,j} \leq C_{\mathcal{D}}\tilde{g}_j$. Also, by (5.1)–(5.2), and since $\ell_0 \geq 1$ and $k_0 \leq 1$ (as chosen in [29, Section 1.7.1]),

$$\begin{aligned} |\lambda_x|h_j^p h_{\sigma,j} &= |\lambda_x|\ell_j^p\ell_{\sigma,j}^p(h_j/\ell_j)^p(h_{\sigma,j}/\ell_{\sigma,j}) \\ &= |\lambda_x|\ell_j^p\ell_{\sigma,j}^p(k_0/\ell_0)^p\tilde{g}_j^{-p/4}\tilde{g}_j^{p/4-1} \\ &\leq C_{\mathcal{D}}\tilde{g}_j(k_0/\ell_0)^p\tilde{g}_j^{-1} \leq C_{\mathcal{D}}k_0^p. \end{aligned} \quad (7.5)$$

This fulfills the required bound on $\epsilon_{V,j}$ of [28, Proposition 1.4].

7.1.5 Stability estimate: case of $p \geq 4$

For $p > 2$, the proof of [28, Proposition 5.1] must be modified. In particular, for $p \geq 4$, we must justify placing such a large power in the exponent, as this appears to make the expectation of e^{-V} divergent since the measure provides only exponentially quadratic decay. Justification is possible because functions of σ_a and σ_b are equivalent to second-order polynomials, by definition of the quotient space in (2.29)–(2.31). Because of this, the placement of the observables in the exponent is an option that superficially appears worse than it actually is.

In more detail, by definition of \mathcal{N} , we have $e^{\lambda_a\sigma_a\bar{\phi}_a^p} = 1 + \lambda_a\sigma_a\bar{\phi}_a^p$. Therefore,

$$\begin{aligned} \|e^{\lambda_a\sigma_a\bar{\phi}_a^p}\|_{T_\phi} &\leq 1 + |\lambda_a|\mathfrak{h}_\sigma\|\bar{\phi}_a^p\|_{T_\phi} \leq 1 + |\lambda_x|\mathfrak{h}_\sigma\mathfrak{h}^p(1 + \|\phi\|_\Phi)^{2p} \\ &\leq e^{2p(|\lambda_x|\mathfrak{h}_\sigma\mathfrak{h}^p)^{1/p}(1 + \|\phi\|_\Phi^2)}, \end{aligned} \quad (7.6)$$

where in the second inequality we used [26, Proposition 3.10], and in the third we used the elementary fact (see [28, Lemma 5.2]) that $1 + u^p(1+x)^{2p} \leq e^{2pu(1+x^2)}$ for any $x, u > 0$ and $p \geq \max\{1, u\}$, with the choice $u = (|\lambda_x|\mathfrak{h}_\sigma\mathfrak{h}^p)^{1/p}$. This modification permits the proof of [28, Proposition 5.1] to proceed as it is otherwise written.

7.2 Modification to [29, Map 6]

For the analysis of Map 6 in [29, Section 6.2], we must estimate the increments $\delta q_a, \delta q_b, \delta t_a, \delta t_b$, and δu that arise in R_+ . The discussion of δu provided there holds without change here. There is a small modification to the treatment of $\delta q_a, \delta q_b$, which we discuss first, and $\delta t_a, \delta t_b$ are new here. We use the notation of [29, Section 6.2].

Let $x = a, b$. It suffices to show that $\|\delta q_x \sigma_a \sigma_b\|_{T_0} \prec 1$, and for this we may assume that $j \geq j_{ab}$. In this case, $\lambda_x = \lambda_{j_{ab}, x}$ and λ_x is not updated by Q . By [13, Proposition 6.1], for $m^2 \in \mathbb{I}_j$, $|C_{j;xy}| \leq cL^{-2(j-1)}$. From this we conclude that $\delta[w_{ab}^p] \prec L^{-2pj} \prec \ell_j^{2p}$. Therefore,

$$\delta q_x = p! \lambda_a \lambda_b \delta[w_{ab}^p] \prec \lambda_a \lambda_b \ell_j^{2p}. \quad (7.7)$$

Since $V \in \mathcal{D}_j$, we have $|\lambda_x| \leq C_{\mathcal{D}} \tilde{g}_j \ell_j^{-p} \ell_{\sigma, j}^{-1}$. Therefore,

$$\|\delta q_x \sigma_a \sigma_b\|_{T_0} = |\delta q_x| \mathfrak{h}_{\sigma, j}^2 \prec \tilde{g}_j^2 (\mathfrak{h}_{\sigma, j} / \ell_{\sigma, j})^2. \quad (7.8)$$

Since the right-hand side is \tilde{g}_j^2 for $\mathfrak{h} = \ell$, and is $\tilde{g}_j^{2p/4}$ for $\mathfrak{h} = h$, this is sufficient.

Finally, δt_x only arises for $n \geq 1$ and $p = 2$, which we assume in the following sketch. It suffices to show that $|\delta t_x|_{\mathfrak{h}_\sigma} \prec 1$. By (3.40),

$$\delta t_x = t_{\text{pt}, x}(V - Q) - t_x = \mathbb{1}_{n \geq 1} \hat{\lambda}_x (e^+ \cdot h) \hat{\zeta}, \quad (7.9)$$

where

$$\hat{\zeta} = \left(C_{0,0} (1 - \mathbb{1}_{j+1 < j_{ab}} 2\hat{\nu} w^{(1)}) + \mathbb{1}_{j+1 < j_{ab}} \hat{\nu}^+ \delta[w^{(2)}] + \mathbb{1}_{j+1 \geq j_{ab}} \delta[\hat{\nu} w^{(2)}] \right), \quad (7.10)$$

with $\hat{\lambda}_x, \hat{\nu}$ the relevant coupling constants of $V - Q$. Thus, $\hat{\lambda}_x = \lambda_x - \lambda_{x,Q}$ and $\hat{\nu} = \nu - \nu_Q$, with $\lambda_{x,Q}$ and ν_Q from Q . As above, we have $C_{00} \prec \ell_j^2$ and $|\lambda_x| \leq C_{\mathcal{D}} \tilde{g}_j \ell_j^{-2} \ell_{\sigma, j}^{-1}$. As in [29, (1.43)], we define

$$\bar{\epsilon}_j = \begin{cases} \chi_j^{1/2} \tilde{g}_j & (\mathfrak{h} = \ell) \\ \chi_j^{1/2} \tilde{g}_j^{1/4} & (\mathfrak{h} = h). \end{cases} \quad (7.11)$$

In the setting of Map 6, we have $|\lambda_{x,Q}| \mathfrak{h}^2 \mathfrak{h}_\sigma \prec \bar{\epsilon}$. The largest term on the right-hand side of (7.10) is the first one, and its contribution to $|\delta t_x|_{\mathfrak{h}_\sigma}$ is bounded by a multiple of

$$|\hat{\lambda}_x| \ell_j^2 \mathfrak{h}_{\sigma, j} \prec (\tilde{g}_j \ell_j^{-2} \ell_{\sigma, j}^{-1} + \bar{\epsilon}_j \mathfrak{h}_j^{-2} \mathfrak{h}_{\sigma, j}^{-1}) \ell_j^2 \mathfrak{h}_{\sigma, j} \prec \tilde{g}_j^{1/4} + \bar{\epsilon}_j \tilde{g}_j^{-1/2}, \quad (7.12)$$

for both $\mathfrak{h} = \ell$ or $\mathfrak{h} = h$ (recall (5.1)–(5.2)). This is sufficient.

7.3 Renormalisation and reduced symmetry

As discussed in Section 4.2, for $n \geq 2$ the $O(n)$ symmetry can be reduced by choice of h . To handle this, we replaced the definition of the space \mathcal{K} in [29, Definition 1.7] by the adapted version in Definition 4.5. With Definition 4.5, we can prove that if $h \in E^\pm$, and if $U \in \mathcal{V}_h$ and $K \in \mathcal{K}(h)$ obey appropriate estimates, then under the renormalisation group map it is also the case that $V_+ \in \mathcal{V}_h$ and $K_+ \in \mathcal{K}_+(h)$. This is the content of the following proposition, in which we place more prominence than usual on h in the notation.

Proposition 7.1. *The renormalisation group map of [29, Section 1.8] obeys $(R_+, K_+) : \mathbb{D}(\tilde{s}, h) \times \tilde{\mathbb{I}}_+(\tilde{m}^2) \rightarrow \mathcal{V}_h^{(1)} \times \mathcal{W}_+(\tilde{s}_+, h)$.*

The proof of Proposition 7.1 is organised as follows. In Section 7.3.1, we prove elementary properties of h -factorisability. In Section 7.3.2, we prove that R_+ maps into $\mathcal{V}_h^{(1)}$. In Section 7.3.3, we prove that K_+ maps into $\mathcal{W}_+(\tilde{s}_+)$.

7.3.1 Elementary properties of h -factorisability

Lemma 7.2. *Let $n \geq 1$ and $h \in \mathbb{R}^n$. If $F, K \in \mathcal{N}_{h\text{-fac}}$, and if $\pi_\emptyset F$ and $\pi_\emptyset K$ are $S(n)$ -invariant, then $FK \in \mathcal{N}_{h\text{-fac}}$ with $(FK)_\alpha^* = (\pi_\emptyset F)K_\alpha^* + F_\alpha^*(\pi_\emptyset K)$ for $\alpha = a, b$.*

Proof. We write $F_\emptyset = \pi_\emptyset F$ and $K_\emptyset = \pi_\emptyset K$. Since we work in a quotient space with $\sigma_\alpha^2 = 0$,

$$\pi_\alpha FK = F_\emptyset(\pi_\alpha K) + (\pi_\alpha F)K_\emptyset = \sigma_\alpha([F_\emptyset K^* + F^* K_\emptyset] \cdot h), \quad (7.13)$$

so the first requirement of Definition 4.3 holds with $(FK)^*$ as indicated. Secondly, by the hypotheses on F_\emptyset and K , for $P \in S(n)$,

$$\begin{aligned} (P(F_\emptyset K_\alpha^*))(\varphi) &= (F_\emptyset(PK_\alpha^*))(\varphi) = F_\emptyset(\varphi)(PK_\alpha^*)(\varphi) \\ &= F_\emptyset(P\varphi)K_\alpha^*(P\varphi) = (F_\emptyset K_\alpha^*)(P\varphi). \end{aligned} \quad (7.14)$$

The $F^* K_\emptyset$ term is similar, and the proof is complete. \blacksquare

Lemma 7.3. *Let $n \geq 1$, $X \subset \Lambda$, and $F \in \mathcal{N}_{h\text{-fac}}$. Then $\text{Loc}_X F \in \mathcal{N}_{h\text{-fac}}$ with $(\text{Loc}_X F)_\alpha^* = \text{Loc}_X F_\alpha^*$. Also, $\mathbb{E}\theta F \in \mathcal{N}_{h\text{-fac}}$ with $(\mathbb{E}\theta F)_\alpha^* = \mathbb{E}\theta F_\alpha^*$. Here $\text{Loc}_X F_\alpha^*$ and $\mathbb{E}\theta F_\alpha^*$ are defined component-wise.*

Proof. The statement has content only for $n \geq 2$, so we write the proof for this case. Since Loc_X commutes with π_α and is linear,

$$\pi_\alpha \text{Loc}_X F = \text{Loc}_X \pi_\alpha F = \text{Loc}_X \sigma_\alpha(F_\alpha^* \cdot h) = \sigma_\alpha(\text{Loc}_X F_\alpha^* \cdot h). \quad (7.15)$$

The invariance under permutations follows easily.

Again by linearity, $\pi_\alpha \mathbb{E}\theta F = \mathbb{E}\theta \pi_\alpha F = \mathbb{E}\theta \sigma_\alpha(F_\alpha^* \cdot h) = \sigma_\alpha(\mathbb{E}\theta F_\alpha^* \cdot h)$. For the invariance under permutations $P \in S(n)$ of the fields, we use

$$\begin{aligned} (P(\mathbb{E}\theta F_\alpha^*))(\varphi) &= \mathbb{E}\theta(PF_\alpha^*)(\varphi) = \mathbb{E}(PF_\alpha^*)(\varphi + \zeta) = \mathbb{E}F_\alpha^*(P(\varphi + \zeta)) \\ &= \mathbb{E}F_\alpha^*(P\varphi + \zeta) = (\mathbb{E}\theta F_\alpha^*)(P\varphi), \end{aligned} \quad (7.16)$$

where ζ is the integration variable, and where the fourth equality follows by making the change of variables $\zeta \mapsto P\zeta$ (with Jacobian equal to 1) in the integral. \blacksquare

The following lemma shows that I_j and related quantities are Euclidean covariant (recall Definition 4.1) and inherit h -factorisability from V .

Lemma 7.4. *Let $V \in \mathcal{V}_h$, $X \in \mathcal{P}_j$, and $x \in \Lambda$. Each of $W_j(V, X)$, $I_j(V, X)$, $P_{j,x}(V)$ and $V_{\text{pt},x}(V)$ is in $\mathcal{N}_{h\text{-fac}}$. Each of $\pi_\emptyset W_j(V)$, $\pi_\emptyset I_j(V)$ (as functions of $X \in \mathcal{P}_j$), and $\pi_\emptyset P_j(V)$ and $\pi_\emptyset V_{\text{pt}}(V)$ (as functions of $x \in \Lambda$) is Euclidean covariant.*

Proof. Let $A \in \mathcal{N}_{h\text{-fac}}$ be a polynomial in the fields, and let $\alpha = a, b$. Then $\pi_\alpha A = \sigma_\alpha(A_\alpha^* \cdot h)$, and we can assume that every component of A_α^* is a polynomial. Recall the definition of \mathcal{L}_C in (3.14). Note that $\pi_\alpha \mathcal{L}_C A = \sigma_\alpha(\mathcal{L}_C A_\alpha^* \cdot h)$. Let $P \in S(n)$ be a permutation matrix. Since \mathcal{L}_C acts

component-wise, $P\mathcal{L}_C A_\alpha^* = \mathcal{L}_C P A_\alpha^*$, and hence, since $A \in \mathcal{N}_{h\text{-fac}}$, $(P\mathcal{L}_C A_\alpha^*)(\varphi) = (\mathcal{L}_C P A_\alpha^*)(\varphi) = (\mathcal{L}_C A_\alpha^*)(P\varphi)$. This shows that $\mathcal{L}_C A \in \mathcal{N}_{h\text{-fac}}$. Consequently,

$$e^{\pm \mathcal{L}_C} A = \sum_{k=0}^{\deg(A)} \frac{(\pm 1)^k}{k!} \mathcal{L}_C^k A \in \mathcal{N}_{h\text{-fac}}. \quad (7.17)$$

Let $V \in \mathcal{V}_h$ and $X \in \mathcal{P}_j$. Then $V \in \mathcal{N}_{h\text{-fac}}$ by definition and every component of V_α^* is a polynomial. Using Lemmas 7.2–7.3 and the above observations concerning \mathcal{L}_C , we see from (3.23)–(3.24) that $F_{\pi, C}(V, V)$ is h -factorisable, as are $W_j(V, X)$, $I_j(V, X)$, $P_j(V, X)$ and $V_{\text{pt}}(V, X)$.

The Euclidean covariance is a consequence of the definitions, the Euclidean invariance of w_j , and the Euclidean covariance property $A(\pi_\emptyset \text{Loc}_X K) = \pi_\emptyset \text{Loc}_{AX}(AK)$ of [27, Proposition 1.9]. ■

7.3.2 Range of R_+

The following proposition gives the R_+ part of Proposition 7.1.

Proposition 7.5. *Let $h \in E^\pm$. If $(U, K) \in \mathbb{D}(\tilde{s})$ and $m^2 \in \tilde{\mathbb{I}}_+(\tilde{m}^2)$, then $R_+(U, K) \in \mathcal{V}_h^{(1)}$.*

The main step in the proof of Proposition 7.5 is provided by Lemma 7.7 below, which in turn relies on Lemma 7.6. For the latter, we observe that the linear span of the permutation subgroup $S(n)$ consists of the set $\bar{S}(n)$ of $n \times n$ matrices whose row and column sums are all equal. Given a set Z of matrices, we write $Z' = \{B : AB = BA \text{ for all } A \in Z\}$ for its commutant. Recall the set of matrices $M_2(n)$ from Definition 3.1. The following lemma states that $\bar{S}(n)$ and $M_2(n)$ are each other's commutant; we omit the elementary proof.

Lemma 7.6. *For $n \geq 1$, $M_2'(n) = \bar{S}(n)$ and $\bar{S}'(n) = M_2(n)$.*

The proof of the following lemma uses the fact that

$$T(\text{Loc}_X F) = \text{Loc}_X(TF) \quad \text{for any } n \times n \text{ matrix } T \text{ and } F \in \mathcal{N}. \quad (7.18)$$

A proof of (7.18) is given in [15, Proposition 2.1] for the case $F \in \mathcal{N}_\emptyset$, and the same proof holds for $F \in \mathcal{N}$. Also, it is shown in [27, Sections 1.4, 1.6] that Loc preserves Euclidean invariance, gauge invariance, and supersymmetry.

Lemma 7.7. *Let $X \subset \Lambda$ and $\alpha \in \{a, b, ab\}$. For $n = 0$, $p \geq 1$, and for $n \geq 1$ and $p = 1, 2$,*

$$\text{Loc}_X(\pi_\alpha \mathcal{N}_h) \subset \bigcup_{m \in M_2(n)} \pi_\alpha \mathcal{V}_{mh}(X). \quad (7.19)$$

In particular, if $h \in E^\pm$, the right-hand side of (7.19) becomes simply $\pi_\alpha \mathcal{V}_h(X)$.

Proof. We use properties of Loc from [27]. By (3.20), an element of $\pi_{ab} \mathcal{V}_h$ can be written as $-\sigma_a \sigma_b \frac{1}{2}(q_a \mathbb{1}_{x=a} + q_b \mathbb{1}_{x=b})$ (independent of h). Thus (7.19) follows from our choice $d_+(ab) = 0$. Similarly, for $\alpha = a, b$ and $j \geq j_{ab}$, elements of $\pi_\alpha \text{Loc}_X(\mathcal{N}_h)$ are constant multiples of σ_α . Thus we assume henceforth that $j < j_{ab}$ and consider $\alpha = a, b$. In this case, $d_+(\alpha) = p$.

For $n = 0$ and $p \geq 1$, h plays no role, and in this case $\sigma_a = \sigma^p$ and $\sigma_b = \bar{\sigma}^p$ (recall Definition 4.2). The only $U(1)$ -invariant monomials containing $\sigma_a = \sigma^p$ or $\sigma_b = \bar{\sigma}^p$, and with dimension at most p , are $\{\sigma^p \bar{\phi}^p, \bar{\sigma}^p \phi^p\}$. Since Loc_X preserves $U(1)$ invariance, this implies that, as required,

$$\text{Loc}_X(\pi_a \mathcal{N}_h) = \mathbb{1}_{a \in X} \sigma^p \text{span} \{\bar{\phi}_a^p\}, \quad \text{Loc}_X(\pi_b \mathcal{N}_h) = \mathbb{1}_{b \in X} \bar{\sigma}^p \text{span} \{\phi_b^p\}. \quad (7.20)$$

The appearance of the indicator functions on the right-hand sides of (7.20) follows directly from the definition of Loc in [27, Definition 1.17].

For $n \geq 1$ and $p = 1, 2$, since $d_+(\alpha) = p$,

$$\text{Loc}_X(\pi_\alpha \mathcal{N}) \subset \mathbb{1}_{\alpha \in X} \sigma_\alpha \text{span} \{1, \varphi_\alpha^i \mid 1 \leq i \leq n\} \quad (p = 1), \quad (7.21)$$

$$\text{Loc}_X(\pi_\alpha \mathcal{N}) \subset \mathbb{1}_{\alpha \in X} \sigma_\alpha \text{span} \{1, \varphi_\alpha^i, \varphi_\alpha^i \varphi_\alpha^k, \nabla \varphi_\alpha^i \mid 1 \leq i, k \leq n\} \quad (p = 2), \quad (7.22)$$

where the superscripts on φ_α indicate components.

For the case $p = 2$, it follows from (7.18) that the $R(n)$ -invariance of \mathcal{N}_h is preserved by Loc_X . The linear, mixed quadratic, and gradient monomials from (7.22) are not invariant under replacement of one component of φ_α by its negative, and thus are not in $\text{Loc}_X(\pi_\alpha \mathcal{N}_h)$ when $p = 2$. Therefore, for both $p = 1$ and $p = 2$,

$$\text{Loc}_X(\pi_\alpha \mathcal{N}_h) \subset \mathbb{1}_{\alpha \in X} \sigma_\alpha \text{span} \{1, (\varphi_\alpha^1)^p, \dots, (\varphi_\alpha^n)^p\}. \quad (7.23)$$

By Definition 4.4, if $F \in \mathcal{N}_h$ then $\pi_\alpha F = \sigma_\alpha(F_\alpha^* \cdot h)$, with $F^* \in (\pi_\emptyset \mathcal{N})^n$ such that $(PF_\alpha^*)(\varphi) = F_\alpha^*(P\varphi)$ for all $P \in S(n)$. Each component of $\text{Loc}_X F_\alpha^*$ lies in $\text{span} \{1, (\varphi_\alpha^1)^p, \dots, (\varphi_\alpha^n)^p\}$, by (7.23). Therefore, there exist an $n \times n$ matrix m_α and a vector $v_\alpha \in \mathbb{R}^n$ such that $\text{Loc}_X F_\alpha^* = m_\alpha \varphi_\alpha^p + v_\alpha$. With Lemma 7.3, this implies that

$$P(m_\alpha \varphi_\alpha^p + v_\alpha) = m_\alpha P \varphi_\alpha^p + v_\alpha \quad (7.24)$$

for every $P \in S(n)$, from which we conclude that $Pv_\alpha = v_\alpha$ and $Pm_\alpha = m_\alpha P$ for every $P \in S(n)$. The first of these conclusions implies that $v_\alpha = s_\alpha e^+$ for some $s_\alpha \in \mathbb{R}$ (with the vector e^+ of (3.30)), and by Lemma 7.6 the second implies that $m_\alpha \in M_2(n)$. Since $m^T = m$ for $m \in M_2(n)$,

$$\begin{aligned} \text{Loc}_X \pi_\alpha F &= \mathbb{1}_{\alpha \in X} \sigma_\alpha (m_\alpha \varphi_\alpha^p \cdot h + s_\alpha e^+ \cdot h) \\ &= \mathbb{1}_{\alpha \in X} \sigma_\alpha (\varphi_\alpha^p \cdot (m_\alpha h) + s_\alpha e^+ \cdot h). \end{aligned} \quad (7.25)$$

The right-hand side lies in $\pi_\alpha \mathcal{V}_{m_\alpha h}(X)$ (with $t_\alpha = s_\alpha e^+ \cdot h$), and this completes the proof. \blacksquare

Proof of Proposition 7.5. Let h be in one of the eigenspaces E^\pm of the matrices in $M_2(n)$ (and $h = 1$ if $n = 0$). The definition of R_+ is given in (4.9)–(4.10). It is already established in [29, Section 2.1] that $\pi_\emptyset R_+ \in \pi_\emptyset \mathcal{V}_h^{(1)}$ (for this h plays no role). Thus we concentrate on $\pi_\alpha R_+$ for $\alpha \in \{a, b, ab\}$. Note that the superscript in $\mathcal{V}_h^{(1)}$ plays no role in these observable subspaces.

By assumption, $U \in \mathcal{V}_h^{(0)} \subset \mathcal{V}_h$, and by Proposition 3.2, $V_{\text{pt}} : \pi_\alpha \mathcal{V}_h \rightarrow \pi_\alpha \mathcal{V}_{h_{\text{pt}}} = \pi_\alpha \mathcal{V}_h$ (with the last equality due to $h \in E^\pm$). Thus, by definition of R_+ , it suffices to show that the polynomial Q defined by (4.9) obeys $\pi_\alpha Q \in \pi_\alpha \mathcal{V}_h$. By definition of Q , to prove that $\pi_\alpha Q \in \pi_\alpha \mathcal{V}_h$ it suffices to prove that $\text{Loc}_X : \pi_\alpha \mathcal{N}_h \rightarrow \pi_\alpha \mathcal{V}_h$, because $K(Y)I(Y, V)^{-1} \in \mathcal{N}_h$ by Lemma 7.4 and because $K(Y) \in \mathcal{N}_h$ since $K \in \mathcal{K}(h)$ (recall Definition 4.5). This last requirement is provided by Lemma 7.7, and the proof is complete. \blacksquare

7.3.3 Range of K_+

We now complete the proof of Proposition 7.1, by proving its K_+ part.

We extend the notion of h -factorisation in Definition 4.3 to maps $F : \mathcal{P}_j \rightarrow \mathcal{N}$, as follows. We say that F is h -factorisable if $F(X) \in \mathcal{N}_{h\text{-fac}}$ for all $X \in \mathcal{P}_j$. By Lemma 7.2, if $F, G \in \mathcal{K}$ are h -factorisable, then $F \circ G$ is h -factorisable as well since the $O(n)$ -invariance of $\pi_\emptyset F$ and $\pi_\emptyset G$ is guaranteed by the definition of \mathcal{K} in Definition 4.5.

Proof of Proposition 7.1. By Proposition 7.5, $R_+(U, K) \in \mathcal{V}_h^{(1)}$, so it remains to check that $K_+ \in \mathcal{W}_+(\tilde{s}_+)$. This statement is provided by [29, Theorem 1.11], apart from the requirement that K_+ is h -factorisable, and, if $p = 2$, that K_+ is $R(n)$ -invariant. To check that the map K_+ constructed in [29] is h -factorisable, we recall that the construction is a composition of six maps which produce $K^{(1)}, \dots, K^{(6)} = K_+$. We examine these one by one and show that $K^{(i)} \in \mathcal{N}_{h\text{-fac}}$ implies $K^{(i+1)} \in \mathcal{N}_{h\text{-fac}}$. We omit the simpler proof that K_+ is $R(n)$ -invariant when $p = 2$.

1. According to its construction in [29, Lemma 4.2], $K^{(1)}$ is a polynomial in I, K , and J (see [29, (4.9)]). Since J is given by localised products of I and K (see [29, (4.12)–(4.13)]), it is h -factorisable, and hence so is $K^{(1)}$, by Lemma 7.2.
2. By [29, Lemma 4.3], $K^{(2)}$ is a circle product of $\delta I^{(2)}$ and $K^{(1)}$. Both of these are h -factorisable, and hence so is $K^{(2)}$.
3. The definition of $K^{(3)}$ is given in [29, (5.9)]. All of the quantities on the right-hand side of [29, (5.9)] are h -factorisable, and hence so is $K^{(3)}$.
4. According to its construction in [29, Lemma 5.8], $K^{(4)}$ is a polynomial in $\tilde{I}_{\text{pt}}, K^{(3)}$, and h_{lead} (see [29, (4.9)]). By [29, (5.18)], h_{lead} is a truncated expectation of V 's, so it is h -factorisable by Lemma 7.3, and hence so is $K^{(4)}$.
5. Map 5 replaces $W(V_{\text{pt}})$ by $W(V_+)$. Since both $V_{\text{pt}}(V)$ and V_+ are h -factorisable, so is $K^{(5)}$.
6. The role of Map 6 is to perform summation by parts and to move constant fields out of the circle product. Only the second aspect is different in our present setting, in which [29, (6.24)] becomes replaced by

$$((e^{\delta\zeta} I_{\text{pt}}^+) \circ K^{(5)})(\Lambda) = e^{\delta\zeta(\Lambda)} (I_{\text{pt}}^+ \circ (e^{-\delta\zeta} K^{(5)}))(\Lambda), \quad (7.26)$$

where

$$\delta\zeta(X) = \sum_{x \in X} V_{\text{pt},x}(V - Q)|_{\varphi=0}. \quad (7.27)$$

We have shown above that $V - Q$ and $V_{\text{pt}}(V - Q)$ are h -factorisable, and hence so is $\delta\zeta$. It can then be seen from its definition in [29, (6.21)] that $K^{(6)}$ is h -factorisable.

Since $K_+ = K^{(6)}$ by definition, this completes the proof. ■

A Proof of Proposition 2.2

In this appendix, we prove Proposition 2.2 using ideas from [25], but organise the proof in a more direct manner for our current goal. The particular approach we present here arose in [18], but these

ideas have a long history going back to [71] and including [20, 25, 37, 41, 58, 60, 62]. Proposition 2.2 can be equivalently stated as the identity

$$\int e^{-\sum_{x \in \Lambda} (\tau_{\Delta, x} + g\tau_x^2 + \nu\tau_x)} \bar{\phi}_{a_1} \cdots \bar{\phi}_{a_p} \phi_{b_1} \cdots \phi_{b_p} = \sum_{\sigma \in S_p} \int_{\mathbb{R}_+^p} E_A^N [e^{-I_p(T)} \mathbb{1}_{X(T)=\sigma(B)}] e^{-\nu\|T\|_1} dT, \quad (\text{A.1})$$

where now the i^{th} walk X^i begins at a_i and ends at $\sigma(b_i)$. The proof of (A.1) is based on three different formulas for the Green function $(-\Delta + V)^{-1}$, where V is a complex diagonal matrix whose diagonal entries v_x obey $\text{Re}(v_x) > 0$. The three formulas are presented in the following three lemmas.

Lemma A.1. *Let \mathcal{W}_{ab}^n denote the set of nearest-neighbour n -step paths from a to b . Then*

$$(-\Delta_{\Lambda} + V)_{ab}^{-1} = \sum_{n=0}^{\infty} \sum_{Y \in \mathcal{W}_{ab}^n} \prod_{j=0}^n \frac{1}{2d + v_{Y_j}}. \quad (\text{A.2})$$

Proof. We write $-\Delta = 2d\mathbb{1} - J$ and let $U = 2d\mathbb{1} + V$. Then $(-\Delta + V)^{-1}$ is given by the Neumann series

$$(-\Delta_{\Lambda} + V)^{-1} = (U - J)^{-1} = \left(U(\mathbb{1} - U^{-1}J) \right)^{-1} = \sum_{n=0}^{\infty} (U^{-1}J)^n U^{-1}, \quad (\text{A.3})$$

which converges since $\text{Re}(V) > 0$. The ab matrix element of the right-hand side is the right-hand side of (A.2), and the proof is complete. \blacksquare

Lemma A.2. *Let $X(T)$ be a continuous time simple random walk on Λ with local time $L_T(x)$. Let V be a complex diagonal matrix with entries v_x such that $\text{Re}(v_x) > 0$, then*

$$(-\Delta_{\Lambda} + V)_{ab}^{-1} = \int_{\mathbb{R}_+} E_a^N [e^{-\sum_{x \in \Lambda} v_x L_T(x)} \mathbb{1}_{X(T)=b}] dT. \quad (\text{A.4})$$

Proof. We think of X as a discrete time simple random walk Y with independent and identically distributed $\text{Exp}(2d)$ holding times $(\sigma_i)_{i \geq 0}$. We set $\gamma_j = \sum_{i=0}^j \sigma_i$, and condition on Y to obtain

$$\begin{aligned} & \int E_a [e^{-v \cdot L_T} \mathbb{1}_{X(T)=b}] dT \\ &= \sum_{n=0}^{\infty} \sum_{Y \in \mathcal{W}_{ab}^n} \left(\frac{1}{2d} \right)^n E \left[e^{-\sum_{j=0}^{n-1} v_{Y_j} \sigma_j} \int_{\gamma_{n-1}}^{\gamma_n} e^{-v_{Y_n} (T - \gamma_{n-1})} dT \right] \\ &= \sum_{n=0}^{\infty} \sum_{Y \in \mathcal{W}_{ab}^n} \left(\frac{1}{2d} \right)^n E \left[\left(e^{-\sum_{j=0}^{n-1} v_{Y_j} \sigma_j} \right) \frac{-1}{v_{Y_n}} (e^{-v_{Y_n} \sigma_n} - 1) \right]. \end{aligned} \quad (\text{A.5})$$

Since the σ_i are i.i.d., the expectation factors into a product of $n + 1$ expectations that can each

be evaluated explicitly, with the result that

$$\begin{aligned}
& \int E_a [e^{-v \cdot L_T} \mathbb{1}_{X(T)=b}] dT \\
&= \sum_{n=0}^{\infty} \sum_{Y \in \mathcal{W}_{ab}^n} \left(\frac{1}{2d} \right)^n \left(\prod_{j=0}^{n-1} \frac{2d}{2d + v_{Y_j}} \right) \left(\frac{2d}{2d + v_{Y_n}} - 1 \right) \left(\frac{-1}{v_{Y_n}} \right) \\
&= \sum_{n=0}^{\infty} \sum_{Y \in \mathcal{W}_{ab}^n} \prod_{j=0}^n \frac{1}{2d + v_{Y_j}}.
\end{aligned} \tag{A.6}$$

By Lemma A.1, this completes the proof. \blacksquare

The next lemma uses the complex Gaussian probability measure on \mathbb{C}^Λ with covariance C , defined by

$$d\mu_C = \frac{\det A}{(2\pi i)^M} e^{-\phi A \bar{\phi}} d\bar{\phi} d\phi, \tag{A.7}$$

with $A = C^{-1}$ and $d\bar{\phi} d\phi$ is the Lebesgue measure $d\bar{\phi}_1 d\phi_1 \cdots d\bar{\phi}_\Lambda d\phi_\Lambda$ (see, e.g., [25, Lemma 2.1] for a proof that this measure is properly normalised). The statement that $d\mu_C$ has covariance C means that $\int \bar{\phi}_a \phi_b d\mu_C = C_{ab}$. Integration by parts (see, e.g., [25, Lemma 2.2]) gives the formula

$$\int_{\mathbb{C}^\Lambda} \bar{\phi}_a F e^{-\phi A \bar{\phi}} d\bar{\phi} d\phi = \sum_x C_{ax} \int_{\mathbb{C}^\Lambda} \frac{\partial F}{\partial \phi_x} e^{-\phi A \bar{\phi}} d\bar{\phi} d\phi. \tag{A.8}$$

Lemma A.3. *Let V be a complex diagonal matrix with entries v_x such that $\text{Re}(v_x) > 0$. Let $A = -\Delta_\Lambda + V$ and set $C = A^{-1} = (-\Delta_\Lambda + V)^{-1}$. Then*

$$\sum_{\sigma \in S_p} \prod_{i=1}^p (-\Delta_\Lambda + V)_{a_i b_{\sigma(i)}}^{-1} = \int e^{-\phi A \bar{\phi} - \psi A \bar{\psi}} \bar{\phi}_{a_1} \cdots \bar{\phi}_{a_p} \phi_{b_1} \cdots \phi_{b_p}. \tag{A.9}$$

Proof. By definition,

$$e^{-\psi A \bar{\psi}} = \sum_{n=0}^M \frac{(-1)^n}{n!} (\psi A \bar{\psi})^n = \frac{(-1)^M}{M!} (\psi A \bar{\psi})^M + (\text{forms of deg} < 2M), \tag{A.10}$$

and only the first (top degree) form on the right-hand side can contribute to the integral. Using $\psi A \bar{\psi} = \sum_{x,y} A_{xy} \psi_x \bar{\psi}_y$ and anti-symmetry, we obtain

$$\begin{aligned}
(\psi A \bar{\psi})^M &= \sum_{x_1, y_1} \cdots \sum_{x_M, y_M} A_{x_1 y_1} \cdots A_{x_M y_M} \psi_{x_1} \bar{\psi}_{y_1} \cdots \psi_{x_M} \bar{\psi}_{y_M} \\
&= \sum_{\eta \in S_M} \sum_{\sigma \in S_M} A_{\eta(1)\sigma(1)} \cdots A_{\eta(M)\sigma(M)} \psi_{\eta(1)} \bar{\psi}_{\sigma(1)} \cdots \psi_{\eta(M)} \bar{\psi}_{\sigma(M)} \\
&= M! \sum_{\sigma \in S_M} A_{1\sigma(1)} \cdots A_{M\sigma(M)} \psi_1 \bar{\psi}_{\sigma(1)} \cdots \psi_M \bar{\psi}_{\sigma(M)} \\
&= M! \sum_{\sigma \in S_M} \text{sgn}(\sigma) A_{1\sigma(1)} \cdots A_{M\sigma(M)} \psi_1 \bar{\psi}_1 \cdots \psi_M \bar{\psi}_M \\
&= (-1)^M M! (\det A) \bar{\psi}_1 \psi_1 \cdots \bar{\psi}_M \psi_M,
\end{aligned} \tag{A.11}$$

so the top degree part of $e^{-\psi A \bar{\psi}}$ is $(\det A) \bar{\psi}_1 \psi_1 \cdots \bar{\psi}_M \psi_M$. Since $\bar{\psi}_x \psi_x = \frac{1}{2\pi i} d\bar{\phi}_x d\phi_x$, this gives

$$\int e^{-\phi A \bar{\phi} - \psi A \bar{\psi}} \bar{\phi}_{a_1} \cdots \bar{\phi}_{a_p} \phi_{b_1} \cdots \phi_{b_p} = \int_{\mathbb{C}^\Lambda} \bar{\phi}_{a_1} \cdots \bar{\phi}_{a_p} \phi_{b_1} \cdots \phi_{b_p} d\mu_{\mathbb{C}}. \quad (\text{A.12})$$

We apply the integration by parts formula (A.8) p times to see that the right-hand is equal to the left-hand side of (A.9), and the proof is complete. (The last step is an instance of Wick's Theorem [46].) \blacksquare

Proof of Proposition 2.2. We prove (A.1). First, we define $F : \mathbb{R}^{\Lambda_N} \rightarrow \mathbb{R}$ by

$$F(S) = e^{-\sum_{x \in \Lambda_N} (gS_x^2 + (\nu-1)S_x)} \quad (S \in \mathbb{R}^{\Lambda_N}). \quad (\text{A.13})$$

Then, by the definition given in (2.1) and the fact that $\sum_x L_T(x) = \|T\|_1$, the summand on the right-hand side of (A.1) is equal to

$$\int_{\mathbb{R}_+^p} E_A^N [e^{-I_p(T)} \mathbb{1}_{X(T)=\sigma(B)}] e^{-\nu \|T\|_1} dT = \int_{\mathbb{R}_+^p} E_A^N [F(L_T) \mathbb{1}_{X(T)=\sigma(B)}] e^{-\|T\|_1} dT. \quad (\text{A.14})$$

Also,

$$\int e^{-\sum_{x \in \Lambda} (\tau_{\Delta, x} + g\tau_x^2 + \nu\tau_x)} \bar{\phi}_a^p \phi_b^p = \int F(\tau) e^{-\sum_{x \in \Lambda} (\tau_{\Delta, x} + \tau_x)} \bar{\phi}_a^p \phi_b^p. \quad (\text{A.15})$$

We write F in terms of its Fourier transform \hat{F} as

$$F(S) = \int e^{-i \sum_{x \in \Lambda} r_x S_x} \hat{F}(r) dr. \quad (\text{A.16})$$

With an appropriate argument to justify interchanges of integration (done carefully in [25]), it therefore suffices to show that for all $s_x \in \mathbb{C}$ with $\text{Re}(s_x) > 0$,

$$\int e^{-\sum_{x \in \Lambda} (\tau_{\Delta, x} + s_x \tau_x)} \bar{\phi}_{a_1} \cdots \bar{\phi}_{a_p} \phi_{b_1} \cdots \phi_{b_p} = \sum_{\sigma \in S_p} \int_{\mathbb{R}_+^p} E_A^N [e^{-\sum_{x \in \Lambda} s_x L_T(x)} \mathbb{1}_{X(T)=\sigma(B)}] dT. \quad (\text{A.17})$$

Let V be the diagonal matrix with entries s_x . Since the components of X are independent and identically distributed, the integral on the right-hand side of (A.17) factors with each factor being $(-\Delta_\Lambda + V)_{a_i \sigma(b_i)}^{-1}$ by Lemma A.2. By Lemma A.3, the left-hand side of (A.17) is therefore equal to the right-hand side, and the proof is complete. \blacksquare

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