SOLUTIONS TO MIDTERM #1, MATH 317

1. (9 marks) Answer true or false to the following questions by putting either true or false in the boxes. If the answer is true give a proof or valid reason, and if the answer is false state why.

(a) If \( C \) is a smooth space curve then \( \mathbf{B}, \mathbf{N}, \mathbf{T} \), in that order, is a right hand system of mutually perpendicular unit vectors.

(b) The space curve \( C : x = \cos t, y = \sin^2 t, z = \sin t, -\infty < t < \infty \), is the intersection of the surfaces \( y = z^2 \) and \( x^2 + z^2 = 1 \).

(c) If \( \mathbf{r}(t) \) is a space curve such that \( \mathbf{r}''(t) \) exists then \( \frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t) \).

Solution:

(a) FALSE. \( \mathbf{B}, \mathbf{N}, \mathbf{T} \) is left hand system of mutually perpendicular unit vectors.

(b) TRUE. We can parametrize the circle \( x^2 + z^2 = 1 \) by \( x = \cos t, y = \sin t \) and then the intersection of the cylinder \( x^2 + z^2 = 1 \) with the paraboloid \( y = z^2 \) can be parametrized by \( x = \cos t, y = \sin^2 t, z = \sin t \).

(c) TRUE. \( \frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}'(t) \times \mathbf{r}'(t) + \mathbf{r}(t) \times \mathbf{r}''(t) = \mathbf{r}(t) \times \mathbf{r}''(t) \) since \( \mathbf{v} \times \mathbf{v} = \mathbf{0} \; \forall \; \mathbf{v} \).

2. (9 marks) The following questions require little or no computation. Enter your answers in the boxes, if provided.

(a) If the space curve \( \mathbf{r}(t) \) has the property that \( \mathbf{r}'(t) \) is always perpendicular to \( \mathbf{r}(t) \) show that \( |\mathbf{r}(t)| \) is a constant for all \( t \).

(b) Evaluate the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( \mathbf{F} = \langle xy, yz, zx \rangle \) and \( C \) is the twisted cubic \( \mathbf{r}(t) = \langle t, t^2, t^3 \rangle , 0 \leq t \leq 1 \).

(c) Find the arc length of the curve \( \mathbf{r}(t) = \langle t^2, \sin t - t \cos t, \cos t + t \sin t \rangle , 0 \leq t \leq \pi \).

Solution:

(a) \( \frac{d}{dt}(|\mathbf{r}(t)|^2) = \frac{d}{dt}(\mathbf{r}(t) \cdot \mathbf{r}(t)) = 2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \implies |\mathbf{r}(t)| \) is a constant

(b) \( \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (xydx + yzdy + zx dz) = \int_{t=0}^{t=1} (t^3 + 2t^6 + 3t^6)dt = \frac{1}{4} + \frac{5}{7} = \frac{27}{28} \).

(c) \( L = \int_{t=0}^{t=\pi} \sqrt{(2t)^2 + (\cos t - \cos t + t \sin t)^2 + (-\sin t + \sin t + t \cos t)^2} dt \)

\( = \int_{t=0}^{t=\pi} \sqrt{5t^2} dt = \frac{\sqrt{5\pi^2}}{2} \)
3. (3 marks) Find the velocity, speed and acceleration of a particle whose position at time \( t \) is given by \( \mathbf{r}(t) = \langle \cos t, t, \sin t \rangle \).

Solution: \( \mathbf{v}(t) = \langle -\sin t, 1, \cos t \rangle, \quad \mathbf{v}(t) = \sqrt{2}, \quad \mathbf{a}(t) = \langle -\cos t, 0, -\sin t \rangle \).

4. (5 marks) Find \( \mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t) \), at \( t = 0 \), for the space curve \( \mathbf{r}(t) = \langle e^t, e^t \sin t, e^t \cos t \rangle \).

Solution:

\[
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{3}e^t} < e^t, e^t \sin t + e^t \cos t, e^t \cos t - e^t \sin t > \\
= \frac{1}{\sqrt{3}} < 1, \sin t + \cos t, \cos t - \sin t > \implies \mathbf{T}(0) = \frac{1}{\sqrt{3}} < 1, 1, 1 > \\
\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1}{\sqrt{2}} < 0, \cos t - \sin t, -\sin t - \cos t > \implies \mathbf{N}(0) = \frac{1}{\sqrt{2}} < 0, 1, -1 >
\]

Therefore \( \mathbf{B}(0) = \frac{1}{\sqrt{6}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{j} - \mathbf{k}) = \frac{1}{\sqrt{6}}(-2\mathbf{i} + \mathbf{j} + \mathbf{k}) \).

5. (4 marks) Determine if the vector field \( \mathbf{F}(x, y) = (ye^x + \sin y)i + (e^x + x \cos y + y)j \) is conservative, and if so find all potential functions.

Solution: We start looking for a potential by solving some partial differential equations.

\[
\frac{\partial f}{\partial x} = ye^x + \sin y \implies f(x, y) = ye^x + x \sin y + C(y) \\
\frac{\partial f}{\partial y} = e^x + x \cos y + C'(y) = e^x + x \cos y + y \implies f(x, y) = ye^x + x \sin y + \frac{1}{2}y^2 + c
\]

6. (4 marks) Prove that the vector field \( \mathbf{F} = \frac{-y}{x^2 + y^2}i + \frac{x}{x^2 + y^2}j \), defined on the plane minus the origin, is not conservative.

Solution: If it were conservative then we would have \( \int_C \mathbf{F} \cdot d\mathbf{r} = 0 \), where \( C \) is the unit circle \( x = \cos t, y = \sin t, 0 \leq t \leq 2\pi \). But

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=0}^{y=2\pi} ((-\sin t)(-\sin t) + (\cos t)(\cos t))dt = 2\pi.
\]