SOLUTIONS TO MIDTERM #1, MATH 317

1. (9 marks) Answer true or false to the following questions by putting either true or false in the boxes. If the answer is true give a proof or valid reason, and if the answer is false state why.

   (a) If C is a smooth space curve then $\mathbf{B}, \mathbf{N}, \mathbf{T}$, in that order, is a right hand system of mutually perpendicular unit vectors.

   (b) The space curve $C : x = \cos t, y = \sin^2 t, z = \sin t, -\infty < t < \infty$, is the intersection of the surfaces $y = z^2$ and $x^2 + z^2 = 1$.

   (c) If $\mathbf{r}(t)$ is a space curve such that $\mathbf{r}''(t)$ exists then $\dfrac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$.

   Solution:

   (a) FALSE. $\mathbf{B}, \mathbf{N}, \mathbf{T}$ is left hand system of mutually perpendicular unit vectors.

   (b) TRUE. We can parametrize the circle $x^2 + z^2 = 1$ by $x = \cos t, y = \sin t$ and then the intersection of the cylinder $x^2 + z^2 = 1$ with the paraboloid $y = z^2$ can be parametrized by $x = \cos t, y = \sin^2 t, z = \sin t$.

   (c) TRUE. $\dfrac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}'(t) \times \mathbf{r}'(t) + \mathbf{r}(t) \times \mathbf{r}''(t) = \mathbf{r}(t) \times \mathbf{r}''(t)$ since $\mathbf{v} \times \mathbf{v} = \mathbf{0}\ \forall\ \mathbf{v}$.

2. (9 marks) The following questions require little or no computation. Enter your answers in the boxes, if provided.

   (a) If the space curve $\mathbf{r}(t)$ has the property that $\mathbf{r}(t)$ is always perpendicular to $\mathbf{r}'(t)$ show that $|\mathbf{r}(t)|$ is a constant for all t.

   (b) Evaluate the line integral $\displaystyle\int_C \mathbf{F} \bullet d\mathbf{r}$, where $\mathbf{F} =< xy, yz, zx >$ and C is the twisted cubic $\mathbf{r}(t) =< t, t^2, t^3 >,\ 0 \le t \le 1$.

   (c) Find the arc length of the curve $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t \sin t \rangle\ ,\ 0 \le t \le \pi$.

   solution:

   (a) $\dfrac{d}{dt}\left(|\mathbf{r}(t)|^2\right) = \dfrac{d}{dt}\left(\mathbf{r}(t) \bullet \mathbf{r}(t)\right) = 2\mathbf{r}(t) \bullet \mathbf{r}'(t) = 0 \Longrightarrow |\mathbf{r}(t)|$ is a constant

   (b) $\displaystyle\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C (xydx + yzdy + zxdz) = \int_{t=0}^{t=1} (t^3 + 2t^6 + 3t^6)dt = \frac{1}{4} + \frac{5}{7} = \frac{27}{28}.$

   (c)

$$\begin{aligned} L &= \int_{t=0}^{t=\pi} \sqrt{(2t)^2 + (\cos t - \cos t + t \sin t)^2 + (-\sin t + \sin t + t\cos t)^2}dt \\ &= \int_{t=0}^{t=\pi} \sqrt{5t^2}dt = \frac{\sqrt{5}\pi^2}{2} \end{aligned}$$

3. (3 marks) Find the velocity, speed and acceleration of a particle whose position at time t is given by $\mathbf{r}(t) =< \cos t, t, \sin t >$.

   Solution: $\mathbf{v}(t) =< -\sin t, 1, \cos t >,\ v(t) = \sqrt{2},\ \mathbf{a}(t) =< -\cos t, 0, -\sin t >$.

4. (5 marks) Find $\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)$, at $t = 0$, for the space curve $\mathbf{r}(t) = \langle e^t, e^t \sin t, e^t \cos t\rangle$.

   Solution:

$$\begin{aligned}
\mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{3}e^t} < e^t, e^t \sin t + e^t \cos t, e^t \cos t - e^t \sin t > \\
&= \frac{1}{\sqrt{3}} < 1, \sin t + \cos t, \cos t - \sin t > \Longrightarrow \mathbf{T}(0) = \frac{1}{\sqrt{3}} < 1, 1, 1 > \\
\mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{1}{\sqrt{2}} < 0, \cos t - \sin t, -\sin t - \cos t > \Longrightarrow \mathbf{N}(0) = \frac{1}{\sqrt{2}} < 0, 1, -1 >
\end{aligned}$$

   Therefore $\mathbf{B}(0) = \frac{1}{\sqrt{6}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (\mathbf{j} - \mathbf{k}) = \frac{1}{\sqrt{6}}(-2\mathbf{i} + \mathbf{j} + \mathbf{k})$.

5. (4 marks) Determine if the vector field $\mathbf{F}(x, y) = (ye^x + \sin y)\mathbf{i} + (e^x + x \cos y + y)\mathbf{j}$ is conservative, and if so find **all** potential functions.

   Solution: We start looking for a potential by soving some partial differential equations.

$$\begin{aligned}
\frac{\partial f}{\partial x} &= ye^x + \sin y \Longrightarrow f(x, y) = ye^x + x \sin y + C(y) \\
\frac{\partial f}{\partial y} &= e^x + x \cos y + C'(y) = e^x + x \cos y + y \Longrightarrow f(x, y) = ye^x + x \sin y + \frac{1}{2}y^2 + c
\end{aligned}$$

6. (4 marks) Prove that the vector field $\mathbf{F} = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}$, defined on the plane minus the origin, is not conservative.

   Solution: If it were conservative then we would have $\int_C \mathbf{F} \bullet d\mathbf{r} = 0$, where C is the unit circle $x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$. But

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_{t=0}^{y=2\pi} ((-\sin t)(-\sin t) + (\cos t)(\cos t))dt = 2\pi.$$