SOLUTIONS TO HOMEWORK ASSIGNMENT # 4

1. Evaluate the following line integrals using Green's theorem:

   (a) $\oint_C ydx - xdy$, where C is the circle $x^2+y^2 = a^2$ oriented in the clockwise direction.

   (b) $\oint_C (y+x)dx + (x+\sin y)dy$, where C is any simple closed smooth curve joining the origin to itself.

   (c) $\oint_C (y - \ln(x^2+y^2))dx + (2\arctan\frac{y}{x})dy$, where C is the positively oriented circle $(x-2)^2+(y-3)^2 = 1$.

   Solution: Let D be the region bounded by C.

   (a) $\oint_C ydx - xdy = -\int\int_D -2dA = 2\pi a^2$.

   (b) $\oint_C (y+x)dx + (x+\sin y)dy = \int\int_D 0\ dA = 0$.

   (c)

$$\oint_C (y - \ln(x^2+y^2))dx + (2\arctan\frac{y}{x})dy$$
$$= \int\int_D \left(2\frac{-yx^{-2}}{1+(y/x)^2} - \left(1 - \frac{2y}{x^2+y^2}\right)\right) dA = -\text{area}(D) = -\pi$$

2. Evaluate $\oint (x^2-xy)dx + (xy-y^2)dy$, where C is the positively oriented triangle with vertices $(0,0), (1,1), (0,2)$.

   Solution:

$$\oint (x^2-xy)dx + (xy-y^2)dy = \int\int_D (y+x)dA = \int_{x=0}^{x=1}\int_{y=x}^{y=-x+2}(y+x)dydx$$
$$= \int_{x=0}^{x=1}\left((-x+2)^2/2 - x^2/2 + x(-2x+2)\right)dx$$
$$= \int_{x=0}^{x=1}(-2x^2+2)dx = -\frac{2}{3}+2 = \frac{4}{3}$$

3. A paticle starts at $(-2,0)$ and moves along the x-axis to $(2,0)$. Then it moves along the upper part of the circle $x^2+y^2 = 4$ back to $(-2,0)$. Compute the work done on this particle by the force field $\mathbf{F}(x,y) =< x, x^3+3xy^2 >$.

   Solution: The work done is

$$W = \oint_C \mathbf{F}\bullet d\mathbf{r} = \oint xdx + (x^3+3xy^2)dy = \int\int_D (3x^2+3y^2)dA$$
$$= \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=2} 3r^3 drd\theta = 12\pi$$

4. Show that $\oint_C (-ydx + xdy) = x_1y_2 - x_2y_1$, where C is the line segment joining (x_1, y_1) to (x_2, y_2)

   Solution: Parametrize the straight line C by $x = x_1 + t(x_2 - x_1), y = y_1 + t(y_2 - y_1)$, where t goes from $t = 0$ to $t = 1$. Therefore

$$\begin{aligned}
\oint_C -ydx + xdy &= \int_{t=0}^{t=1} -(y_1 + t(y_2 - y_1))(x_2 - x_1)dt \\
&+ \int_{t=0}^{t=1} (x_1 + t(x_2 - x_1))(y_2 - y_1)dt \\
&= -y_1(x_2 - x_1) - \frac{(y_2 - y_1)(x_2 - x_1)}{2} \\
&+ x_1(y_2 - y_1) + \frac{(x_2 - x_1)(y_2 - y_1)}{2} \\
&= -y_1(x_2 - x_1) + x_1(y_2 - y_1) = x_1y_2 - x_2y_1
\end{aligned}$$

5. Find the area of the pentagon with vertices $(0, 0), (2, 1), (2, 3), (1, 4), (-1, 1)$

   Solution: Let C be the positively oriented piecewise smooth curve comprised of the 5 straight line segments $C_1, C_2, \ldots, C_5$, around the boundary of the pentagon, where C_1 joins $(0, 0)$ to $(2, 1)$, C_2 joins $(2, 1)$ to $(2, 3)$, etc. If D is the interior of C, then

$$\begin{aligned}
2 \times \text{area}(D) &= \int_{C_1} (-ydx + xdy) + \int_{C_2} (-ydx + xdy) + \int_{C_3} (-ydx + xdy) + \\
&\int_{C_4} (-ydx + xdy) + \int_{C_5} (-ydx + xdy) = 14 \text{ (arithmetic)} \\
&\Longrightarrow \text{area}(D) = 7
\end{aligned}$$

6. Compute the area under one arch of the cycloid $x = t - \sin t, y = 1 - \cos t$ by using Green's theorem.

   Solution: The area is bounded by the x-axis on the bottom, from $x = 0$ to $x = 2\pi$, and by the cycloid on the top. Let C be the bounding curve, that is the curve consisting of the x-axis traversed from $x = 0$ to $x = 2\pi$, followed by the cycloid going from $t = 2\pi$ to $t = 0$. Let C_1 be the x-axis part of C and let C_2 be the cycloid part. Then

$$\begin{aligned}
2 \times \text{area}(D) &= \oint_C -ydx + xdy = \int_{C_1} -ydx + xdy + \int_{C_2} -ydx + xdy \\
&= \int_{C_2} -ydx + xdy \text{ (since } y = 0 \text{ and } dy = 0 \text{ on } C_1) \\
&= \int_{t=2\pi}^{t=0} (-(1 - \cos t)(1 - \cos t) + (t - \sin t)\sin t)\, dt \\
&= \int_{t=2\pi}^{t=0} (-2 + 2\cos t + t\sin t)dt = 4\pi - t\cos t \Big|_{t=2\pi}^{t=0} + \int_{t=2\pi}^{t=0} \cos t dt = 6\pi \\
&\Longrightarrow \text{area}(D) = 3\pi
\end{aligned}$$

7. Let C be a positively oriented smooth simple closed curve that does not go through the origin, and let D be the region it bounds. Show that

$$\oint_C \frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2} = \begin{cases} 0 & \text{if the origin is outside of } D \\ 2\pi & \text{if the origin is interior to } D \end{cases}$$

Solution: The region D is simply connected. First suppose the origin is outside D. The vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$ is defined throughout D and satisfies $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in D. Therefore $\mathbf{F}$ is conservative on D and $\oint_C \frac{-ydx}{x^2+y^2} + \frac{xdy}{x^2+y^2} = 0$.
Next suppose the origin is interior to D. Then $\exists\, \epsilon > 0$ such that $\{(x,y) \mid x^2+y^2 < \epsilon^2\}$ is completely in D. Let C_ϵ be the circle $x^2+y^2 = \epsilon^2$ oriented in the positive direction. Then

$$\oint_C Pdx + Qdy = \oint_{C_\epsilon} Pdx + Qdy = 2\pi \text{ (this calculation was done in class)}$$

8. Compute the curl of the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, where

   (a) $\mathbf{F}(x,y,z) = \cos xz\ \mathbf{j} - \sin xy\ \mathbf{k}$

   (b) $\mathbf{F}(x,y,z) = \frac{x}{x^2+y^2+z^2}\mathbf{i} + \frac{y}{x^2+y^2+z^2}\mathbf{j} + \frac{z}{x^2+y^2+z^2}\mathbf{k}$.

   Solution: The curl of a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is

$$curl\ \mathbf{F} = det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

   (a) $curl\ \mathbf{F} = (-x\cos xy + x\sin xz)\mathbf{i} + y\cos xy\mathbf{j} - z\sin xz\mathbf{k}$

   (b) By calculation $curl\ \mathbf{F} = \mathbf{0}$.

9. Let $\mathbf{F} = \mathbf{F}(x,y,z) = \frac{x}{(x^2+y^2+z^2)^p}\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^p}\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^p}\mathbf{k}$.

   (a) Compute the divergence of $\mathbf{F}$.

   (b) Find that value of p such that $\nabla\bullet\mathbf{F} = 0$.

   Solution: The divergence of a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is $div\ \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

   (a) $div\ \mathbf{F} = \frac{3(x^2+y^2+z^2)^p - 2p(x^2+y^2+z^2)^p}{(x^2+y^2+z^2)^{2p}} = \frac{3-2p}{(x^2+y^2+z^2)^p}$

   (b) $p = 3/2$.

10. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field on $\mathbb{R}^3$ such thar P, Q, R have continuous second order partials. Then prove that div(curl $\mathbf{F}$) $= 0$.

Solution:

$$div(curl\ \mathbf{F}) \quad = \quad \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0$$

since the mixed partials cancel.