SOLUTIONS TO HOMEWORK ASSIGNMENT # 4

1. Evaluate the following line integrals using Green’s theorem:

   (a) \[ \oint_C y \, dx - x \, dy, \] where \( C \) is the circle \( x^2 + y^2 = a^2 \) oriented in the clockwise direction.

   (b) \[ \oint_C (y + x) \, dx + (x + \sin y) \, dy, \] where \( C \) is any simple closed smooth curve joining the origin to itself.

   (c) \[ \oint_C (y - \ln(x^2 + y^2)) \, dx + (2 \arctan \frac{y}{x}) \, dy, \] where \( C \) is the positively oriented circle \( (x-2)^2 + (y-3)^2 = 1 \).

Solution: Let \( D \) be the region bounded by \( C \).

   (a) \[ \oint_C y \, dx - x \, dy = - \iint_D -2 \, dA = 2\pi a^2. \]

   (b) \[ \oint_C (y + x) \, dx + (x + \sin y) \, dy = \iint_D 0 \, dA = 0. \]

   (c) \[
   \oint_C (y - \ln(x^2 + y^2)) \, dx + (2 \arctan \frac{y}{x}) \, dy
   = \iint_D \left( 2 \frac{-yx^{-2}}{1 + (y/x)^2} - \left( 1 - \frac{2y}{x^2 + y^2} \right) \right) \, dA = \text{area}(D) = -\pi
   
   \]

2. Evaluate \( \oint_C (x^2 - xy) \, dx + (xy - y^2) \, dy \), where \( C \) is the positively oriented triangle with vertices \((0, 0), (1, 1), (0, 2)\).

Solution:

   \[
   \oint_C (x^2 - xy) \, dx + (xy - y^2) \, dy = \iint_D (y + x) \, dA
   = \int_{x=0}^{x=1} \int_{y=x}^{y=x+2} (y + x) \, dy \, dx
   = \int_{x=0}^{x=1} \left( (-x + 2)^2/2 - x^2/2 + x(-2x + 2) \right) \, dx
   = \int_{x=0}^{x=1} (-2x^2 + 2) \, dx = -\frac{2}{3} + 2 = \frac{4}{3}
   
   \]

3. A particle starts at \((-2, 0)\) and moves along the \( x \)-axis to \((2, 0)\). Then it moves along the upper part of the circle \( x^2 + y^2 = 4 \) back to \((-2, 0)\). Compute the work done on this particle by the force field \( \mathbf{F}(x, y) = < x, x^3 + 3xy^2 > \).

Solution: The work done is

   \[
   W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D x \, dx + (x^3 + 3xy^2) \, dy = \iint_D (3x^2 + 3y^2) \, dA
   = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=2} 3r^3 \, dr \, d\theta = 12\pi
   
   \]
4. Show that \( \int_C (-ydx + xdy) = x_1y_2 - x_2y_1 \), where \( C \) is the line segment joining \((x_1, y_1)\) to \((x_2, y_2)\).

Solution: Parametrize the straight line \( C \) by \( x = x_1 + t(x_2 - x_1), y = y_1 + t(y_2 - y_1), \) where \( t \) goes from \( t = 0 \) to \( t = 1 \). Therefore

\[
\int_C -ydx + xdy = \int_{t=0}^{t=1} -(y_1 + t(y_2 - y_1))(x_2 - x_1)dt
+ \int_{t=0}^{t=1} (x_1 + t(x_2 - x_1))(y_2 - y_1)dt
= -y_1(x_2 - x_1) - \frac{(y_2 - y_1)(x_2 - x_1)}{2}
+ x_1(y_2 - y_1) + \frac{(x_2 - x_1)(y_2 - y_1)}{2}
= -y_1(x_2 - x_1) + x_1(y_2 - y_1) = x_1y_2 - x_2y_1
\]

5. Find the area of the pentagon with vertices \((0, 0), (2, 1), (2, 3), (1, 4), (-1, 1)\).

Solution: Let \( C \) be the positively oriented piecewise smooth curve comprised of the 5 straight line segments \( C_1, C_2, \ldots, C_5 \), around the boundary of the pentagon, where \( C_1 \) joins \((0, 0)\) to \((2, 1)\), \( C_2 \) joins \((2, 1)\) to \((2, 3)\), etc. If \( D \) is the interior of \( C \), then

\[
2 \times \text{area}(D) = \int_{C_1} (-ydx + xdy) + \int_{C_2} (-ydx + xdy) + \int_{C_3} (-ydx + xdy) + \int_{C_4} (-ydx + xdy) + \int_{C_5} (-ydx + xdy) = 14 \text{ (arithmetic)}
\]

\[
\implies \text{area}(D) = 7
\]

6. Compute the area under one arch of the cycloid \( x = t - \sin t, y = 1 - \cos t \) by using Green’s theorem.

Solution: The area is bounded by the \( x \)-axis on the bottom, from \( x = 0 \) to \( x = 2\pi \), and by the cycloid on the top. Let \( C \) be the bounding curve, that is the curve consisting of the \( x \)-axis traversed from \( t = 0 \) to \( t = 2\pi \), followed by the cycloid going from \( t = 2\pi \) to \( t = 0 \). Let \( C_1 \) be the \( x \)-axis part of \( C \) and let \( C_2 \) be the cycloid part. Then

\[
2 \times \text{area}(D) = \oint_C -ydx + xdy = \int_{C_1} -ydx + xdy + \int_{C_2} -ydx + xdy
= \int_{C_2} -ydx + xdy \text{ (since } y = 0 \text{ and } dy = 0 \text{ on } C_1)
= \int_{t=0}^{t=2\pi} ((1 - \cos t)(1 - \cos t) + (t - \sin t) \sin t) dt
= \int_{t=2\pi}^{t=0} (-2 + 2 \cos t + t \sin t) dt = 4\pi - t \cos t \bigg|_{t=2\pi}^{t=0} + \int_{t=2\pi}^{t=0} \cos t dt = 6\pi
\]

\[
\implies \text{area}(D) = 3\pi
\]
7. Let $C$ be a positively oriented smooth simple closed curve that does not go through the origin, and let $D$ be the region it bounds. Show that

$$\oint_C -\frac{y \, dx}{x^2 + y^2} + \frac{x \, dy}{x^2 + y^2} = \begin{cases} 0 & \text{if the origin is outside of } D \\ 2\pi & \text{if the origin is interior to } D \end{cases}$$

Solution: The region $D$ is simply connected. First suppose the origin is outside $D$. The vector field $F = P \mathbf{i} + Q \mathbf{j} = -\frac{y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j}$ is defined throughout $D$ and satisfies $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in $D$. Therefore $F$ is conservative on $D$ and $\oint_C -\frac{y \, dx}{x^2 + y^2} + \frac{x \, dy}{x^2 + y^2} = 0$.

Next suppose the origin is interior to $D$. Then $\exists \epsilon > 0$ such that $\{(x, y) \mid x^2 + y^2 < \epsilon^2\}$ is completely in $D$. Let $C_\epsilon$ be the circle $x^2 + y^2 = \epsilon^2$ oriented in the positive direction. Then

$$\oint_C P \, dx + Q \, dy = \oint_{C_\epsilon} P \, dx + Q \, dy = 2\pi \text{ (this calculation was done in class)}$$

8. Compute the curl of the vector field $F = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$, where

(a) $F(x, y, z) = \cos xz \mathbf{j} - \sin xy \mathbf{k}$

(b) $F(x, y, z) = \frac{x}{x^2 + y^2 + z^2} \mathbf{i} + \frac{y}{x^2 + y^2 + z^2} \mathbf{j} + \frac{z}{x^2 + y^2 + z^2} \mathbf{k}$.

Solution: The curl of a vector field $F = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$ is

$$\text{curl } F = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

(a) $\text{curl } F = (-x \cos xy + x \sin xz) \mathbf{i} + y \cos xy \mathbf{j} - z \sin xz \mathbf{k}$

(b) By calculation $\text{curl } F = 0$.

9. Let $F = F(x, y, z) = \frac{x}{(x^2 + y^2 + z^2)^p} \mathbf{i} + \frac{y}{(x^2 + y^2 + z^2)^p} \mathbf{j} + \frac{z}{(x^2 + y^2 + z^2)^p} \mathbf{k}$.

(a) Compute the divergence of $F$.

(b) Find that value of $p$ such that $\nabla \cdot F = 0$.

Solution: The divergence of a vector field $F = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$ is $\text{div } F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

(a) $\text{div } F = \frac{3(x^2 + y^2 + z^2)^p - 2p(x^2 + y^2 + z^2)^{p-1}}{(x^2 + y^2 + z^2)^{2p}} = \frac{3 - 2p}{(x^2 + y^2 + z^2)^p}$

(b) $p = 3/2$. 
10. Let \( \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} \) be a vector field on \( \mathbb{R}^3 \) such that \( P, Q, R \) have continuous second order partials. Then prove that \( \text{div}(\text{curl } \mathbf{F}) = 0 \).

Solution:

\[
\text{div}(\text{curl } \mathbf{F}) = \frac{\partial}{\partial x} \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + \frac{\partial}{\partial y} \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = 0
\]

since the mixed partials cancel.