1. Determine the nature of all singularities of the following functions \( f(z) \).

(a) \( f(z) = \cos \frac{1}{z} \).

(b) \( f(z) = \frac{1}{z^2 \sin z} \).

(c) \( f(z) = \frac{z}{e^{z^2} - 1} \).

Solution:

(a) \( z = 0 \) is the only singularity. It is an essential singularity since the Laurent series expansion about \( z = 0 \),

\[
\cos \frac{1}{z} = 1 - \frac{1}{2!z^2} + \frac{1}{4!z^4} + \cdots,
\]

has infinitely many negative powers of \( z \).

(b) The singularities are \( z = 0 \) and \( z = n\pi, n = \pm 1, \pm 2, \ldots \). The singularity at \( z = 0 \) is a pole of order 3 since \( z = 0 \) is a zero of order 3 of \( z^2 \sin z \). This follows easily from the Maclaurin series about \( z = 0 \):

\[
z^2 \sin z = z^3 - \frac{1}{3!}z^5 + \frac{1}{5!}z^7 + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!}z^{2n+3}.
\]

The singularities \( z = n\pi, n = \pm 1, \pm 2, \ldots \), are simple poles since they are simple zeros of \( z^2 \sin z \).

(c) \( z = 0 \) is a simple pole since

\[
\frac{z}{e^{z^2} - 1} = \frac{z}{z^2 + z^4/2! + z^6/3! + \cdots} = \frac{1}{z + z^3/2! + z^5/3! + \cdots} = \frac{1}{z}g(z)
\]

where \( g(z) \) is analytic at \( z = 0 \) and \( g(0) \neq 0 \). In fact \( g(0) = 1 \), although what’s important is just that \( g(0) \neq 0 \).

The other singularities are the non-zero solutions of \( e^{z^2} = 1 \), that is \( z = \sqrt{2n\pi}i \), where \( n \) is a non-zero integer. They are all simple poles since

\[
\left. \frac{d}{dz} (e^{z^2} - 1) \right|_{z=\sqrt{2n\pi}i} = 2\sqrt{2n\pi}ie^{2n\pi i} = 2\sqrt{2n\pi}i \neq 0.
\]

2. Evaluate the following integrals. In each case the contour is positively oriented.

(a) \( \int_{|z|=R} z^n dz \), where \( n \) is an integer.

(b) \( \int_{|z|=3} \cot zdz \).
(c) \[ \int_{|z|=3} \frac{1}{z \sin z} \, dz. \]

Solution:

(a) Make the substitution \( z = R e^{i \theta} \). Then \( dz = R i e^{i \theta} \, d\theta \) and so

\[ \int_{|z|=R} z^n \, dz = \int_{\theta=0}^{\theta=2\pi} i R^{n+1} e^{-(n+1)i \theta} \, d\theta = i R^{n+1} \int_{\theta=0}^{\theta=2\pi} e^{-(n+1)i \theta} \, d\theta = \left\{ \begin{array}{ll} 2\pi i R^2 & n = 1 \\ 0 & n \neq 1 \end{array} \right. \]

It is obvious \( \int_{\theta=0}^{\theta=2\pi} e^{-(n+1)i \theta} \, d\theta = 2\pi \) if \( n = 1 \). If \( n \neq 1 \) then the Fundamental Theorem of Calculus gives

\[ \int_{\theta=0}^{\theta=2\pi} e^{-(n+1)i \theta} \, d\theta = \frac{e^{-(n+1)i \theta}}{-n+1} \bigg|_{\theta=0}^{\theta=2\pi} = 0 \]

The point to this question is that the function \( f(z) = \bar{z} \) is not analytic, for if it were the Cauchy Integral Theorem would tell us that \( \int_{|z|=R} \bar{z}^n \, dz = 0 \) for \( n \geq 0 \).

(b) This is a straightforward application of the Cauchy Residue Theorem:

\[ \int_{|z|=3} \cot z \, dz = 2\pi i \text{Residue}(\cot z, z = 0) = 2\pi i \frac{z \cos z}{\sin z} \bigg|_{z=0} = 2\pi i. \]

The singularities of \( \cot z = \frac{\cos z}{\sin z} \) are \( z = n\pi, n = 0, \pm 1, \pm 2, \ldots \) They are all simple poles, but only the singularity at \( z = 0 \) is inside the circle \( |z| = 3 \).

(c) The singularities of \( \frac{1}{z \sin z} \) inside the circle \( |z - 1| = 4 \) are \( z = 0 \) and \( z = \pi \). The singularity at \( z = 0 \) is a pole of order 2 since the Laurent series at \( z = 0 \) is

\[ \frac{1}{z \sin z} = \frac{1}{z^2(1 - z^2/3! + z^4/5! - \cdots)} = \frac{1}{z^2} + \frac{1}{6} + \cdots \]

Here we have used the geometric series:

\[ \frac{1}{z \sin z} = \frac{1}{z(z - z^3/3! + z^5/5! - \cdots)} = \frac{1}{z^2(1 - z^2/3! + z^4/5! - \cdots)} \]

\[ = \frac{1}{z^2(1 - (z^2/3! - z^4/5! + \cdots))} \]

\[ = \frac{1}{z^2} \left(1 + (z^2/3! - z^4/5! + \cdots) + (z^2/3! - z^4/5! + \cdots)^2 + \cdots \right) \]

\[ = \frac{1}{z^2} + 1/3! + \text{higher powers of } z \]

Therefore the residue at \( z = 0 \) is 0.
Another way to see this is that

\[
\frac{1}{z \sin z} = \frac{1}{z^2} g(z) \quad \text{where} \quad g(z) = \frac{z}{\sin z}
\]

Now we could expand \( g(z) = z / \sin z \) as a Taylor series about \( z = 0 \). But since \( g(z) \) is an even function it follows that the Taylor series will have the form \( a_0 + a_1 z^2 + a_4 z^4 + \cdots \), and therefore the residue at \( z = 0 \) is 0. We don’t actually have to compute the Taylor series.

The singularity at \( z = \pi \) is a simple pole and therefore the residue at \( z = \pi \) is \( \frac{z - \pi}{z \sin z} \bigg|_{z=\pi} = -1/\pi \). Therefore

\[
\int_{|z|=\pi} \frac{1}{z \sin z} dz = -2\pi.
\]

3. Let \( f(z) \) be the power series \( \sum_{n=0}^{\infty} n^2 z^n \).

(a) Find all \( z \) such that the power series converges.

(b) Find a closed form expression for \( f(z) \).

Solution:

(a) By the ratio test the series converges for \(|z| < 1\) and diverges for \(|z| > 1\). The series diverges for \(|z| = 1\) since the terms \( n^2 z^n \) do not go to 0 as \( n \to \infty \) if \( |z| = 1 \).

(b) Consider the geometric series

\[
\frac{1}{1 - z} = 1 + z + z^2 + z^3 + \cdots
\]

Then

\[
z \frac{d}{dz} \left( \frac{1}{1 - z} \right)^{-1} = z + 2z^2 + 3z^3 + \cdots
\]

Do it one more time:

\[
z + 2^2 z^2 + 3^2 z^3 + \cdots = z \frac{d}{dz} \left( \frac{z}{1 - z} \right)^{-1} = z \frac{d}{dz} \left( \frac{z}{1 - z} \right)^{-2} = \frac{z(1 + z)}{(1 - z)^3}
\]

4. Find all \( z \) such that the power series \( \sum_{n=1}^{\infty} \frac{1}{n^2 z^n} \) converges.

Solution: By the ratio test we see that \( \sum_{n=1}^{\infty} \frac{1}{n^2 z^n} \) converges for \(|z| < 1\) and diverges for \(|z| > 1\). It also converges for \(|z| = 1\) by comparison with the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).

5. Suppose \( f(z) \) is analytic for \(|z| \leq 1\) and \(|f(z)| \leq M\) for \(|z| = 1\), where \( M \) is some constant. Show that \(|f(0)| \leq M\) and \(|f'(0)| \leq M\).
Solution: This follows from a Cauchy Integral Formula and the $ML$ inequality:

$$f(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z} \, dz \implies |f(0)| \leq \frac{1}{2\pi} M 2\pi = M$$

$$f'(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^2} \, dz \implies |f'(0)| \leq \frac{1}{2\pi} M 2\pi = M$$

**Exercise:** What inequalities do you get for $|f^{(n)}(0)|$?

6. Determine if there is a function $f(z)$ which is analytic in some open neighbourhood of the origin and which satisfies the following. If there is such a function find a closed form for it and state where $f(z)$ is analytic.

   (a) $f^{(k)}(0) = k$ for $k \geq 0$.
   (b) $f^{(k)}(0) = (k!)^2$ for $k \geq 0$.
   (c) $f(0) = \pi$ and $f^{(k)}(0) = (-1)^{k+1} 2^k (k-1)!$ for $k \geq 1$.

Solution: In all cases we consider the Maclaurin series $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$.

   (a) $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} z^k = ze^z$. Thus $f(z)$ is entire.

   (b) In this case we would have $f(z) = \sum_{k=0}^{\infty} k! z^k$, which diverges for all $z \neq 0$. Thus there is no such function.

   (c) $f(z) = \pi + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k (k-1)!}{k} z^k = \pi + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} (2z)^k = \pi + \text{Log}(1 + 2z)$. This converges for $|z| < 1/2$.

7. Evaluate the following integrals. In each case the contour is positively oriented.

   (a) $\int_{C_R} \frac{1}{z^2 + z + 1} \, dz$, where $R > 1$ and $C_R$ is the real axis from $-R$ to $R$ together with the upper half of the circle $|z| = R$.

   (b) $\int_{|z|=1} z^2 e^{1/z} \sin(1/z) \, dz$.

Solution:

   (a) The singularities of $f(z) = \frac{1}{z^2 + z + 1}$ occur at the roots of $z^2 + z + 1$. The only root inside the contour $C_R$ is $z = e^{2\pi i/3}$, and it is a simple pole. Thus
\[
\int_{C_R} \frac{1}{z^2 + z + 1} \, dz = 2\pi i \text{Residue} \left( \frac{1}{z^2 + z + 1}, z = e^{2\pi i/3} \right)
= 2\pi i \left. \frac{z - e^{2\pi i/3}}{z^2 + z + 1} \right|_{z=e^{2\pi i/3}}
= 2\pi i \left. \frac{1}{2z + 1} \right|_{z=e^{2\pi i/3}} = \frac{2\pi}{\sqrt{3}}
\]

(b) The only singularity of \(z^2 e^{1/z} \sin(1/z)\) occurs at \(z = 0\), and it is an essential singularity. Therefore the formula for computing the residue at a pole will not work, but we can still compute some of the coefficients in the Laurent series expansion about \(z = 0\):

\[
z^2 e^{1/z} \sin(1/z) = z^2 \left( \frac{1 + 1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \right) \left( \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots \right)
= z + 1 + \frac{1}{3z} + \cdots
\]

\(\implies \text{Residue}(z^2 e^{1/z} \sin(1/z), z = 0) = \frac{1}{3}\)

Therefore \(\int_{|z|=1} z^2 e^{1/z} \sin(1/z) \, dz = \frac{2\pi i}{3}\).

**Exercise:** Read about the Cauchy product in the text.

8. Evaluate \(\int_{0}^{\infty} \frac{x^2 + 1}{x^4 + 1} \, dx\).

**Solution:**

Consider the integral \(\int_{C_R} \frac{1 + z^2}{1 + z^4} \, dz\), where \(R > 0\) and \(C_R\) is the positively oriented contour comprised of the segment of the real axis from \(-R\) to \(R\) and then the upper half of the circle \(|z| = R\). Let \(C'_R\), \(C''_R\) denote the real axis portion and the circular portion resp. Then \(\lim_{R \to \infty} \int_{C''_R} \frac{1 + z^2}{1 + z^4} \, dz = 0\) since the degree of \(z^4 + 1\) is 2 more than the degree of \(z^2 + 1\). The singularities are at the solutions of the equation \(z^4 + 1 = 0\), that is

\(z = e^{\pi i/4}, z = e^{3\pi i/4}, z = e^{5\pi i/4}, z = e^{7\pi i/4}\).

The only singularities in the upper half plane are \(z = e^{\pi i/4}, z = e^{3\pi i/4}\), and they are simple poles. It follows that

\[
\int_{-\infty}^{\infty} \frac{x^2 + 1}{x^4 + 1} \, dx = 2\pi i \left( \text{Residue} \left( \frac{1 + z^2}{1 + z^4}, e^{\pi i/4} \right) + \text{Residue} \left( \frac{1 + z^2}{1 + z^4}, e^{3\pi i/4} \right) \right)
= 2\pi i \left. \left( \frac{(z - e^{\pi i/4})(1 + z^2)}{1 + z^4} \right) \right|_{z=e^{\pi i/4}} + \left. \left( \frac{(z - e^{3\pi i/4})(1 + z^2)}{1 + z^4} \right) \right|_{z=e^{3\pi i/4}}
\]
\[
\begin{aligned}
&= 2\pi i \left( \frac{1+i}{4e^{3\pi i/4}} + \frac{1-i}{4e^{\pi i/4}} \right) = -\frac{\pi i}{2} \left( (1-i)e^{3\pi i/4} + (1+i)e^{\pi i/4} \right) \\
&= -\frac{\pi i}{2} \left( (1-i) \left( \frac{-1+i}{\sqrt{2}} \right) + (1+i) \left( \frac{1+i}{\sqrt{2}} \right) \right) \\
&= -\frac{\pi i}{2} \sqrt{2} \left( -(1-i)^2 + (1+i)^2 \right) = \pi \sqrt{2}
\end{aligned}
\]

Therefore \( \int_0^\infty \frac{x^2+1}{x^4+1} \, dx = \frac{\pi}{\sqrt{2}} \).

**Remarks:** In this calculation we have used the fact that \( \lim_{R \to \infty} \int_{C_R} P(z) Q(z) \, dz = 0 \), where \( P(z), Q(z) \) are polynomials such that \( \deg(Q) \geq \deg(P) + 2 \). See page 322. The basic reason for this is that \( \frac{P(z)}{Q(z)} \) behaves like \( 1/R^d \) on the arc, where \( d = \deg(Q) - \deg(P) \); whereas the arc only has length \( \pi R \). Therefore the \( ML \) inequality guarantees that the integral goes to 0.

9. Evaluate \( \int_{-\pi}^\pi \frac{1}{1 + \sin^2 \theta} \, d\theta \).

Solution:

We make the substitution \( z = e^{i\theta} \). Then \( dz = ie^{i\theta} \, d\theta = iz \, d\theta \), or \( d\theta = \frac{dz}{iz} \). Moreover \( \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - 1/z}{2i} \). Therefore

\[
\int_{-\pi}^\pi \frac{1}{1 + \sin^2 \theta} \, d\theta = \int_{|z|=1} \frac{1}{iz \left( 1 + \left( \frac{z - 1/z}{2i} \right)^2 \right)} \, dz = \int_{|z|=1} \frac{1}{iz \left( 1 + \frac{z^2 - 2 + 1/z^2}{4} \right)} \, dz
\]

\[
= \frac{4}{i} \int_{|z|=1} \frac{1}{z (6 - z^2 - 1/z^2)} \, dz = \frac{4}{i} \int_{|z|=1} \frac{1}{6z^2 - z^4 - 1} \, dz
\]

The singularities occur at solutions of \( z^4 - 6z^2 + 1 = 0 \), that is \( z = \pm \sqrt[4]{3 \pm 2\sqrt{2}} \). All of them are simple poles, but only \( z = \pm \sqrt[4]{3 - 2\sqrt{2}} \) are inside the circle \( |z| = 1 \). Next we compute the residues at these singularities:

\[
\text{Residue} \left( \frac{z}{6z^2 - z^4 - 1} ; z = \sqrt[4]{3 - 2\sqrt{2}} \right) = \frac{z(z - \sqrt[4]{3 - 2\sqrt{2}})}{6z^2 - z^4 - 1} \bigg|_{z=\sqrt[4]{3-2\sqrt{2}}}
\]

\[
= \frac{1}{-4(3 - 2\sqrt{2})^{3/2} + 12\sqrt{3 - 2\sqrt{2}}} = \frac{1}{8\sqrt{2}}
\]

In a similar manner we calculate that

\[
\text{Residue} \left( \frac{z}{6z^2 - z^4 - 1} ; z = -\sqrt[4]{3 - 2\sqrt{2}} \right) = \frac{1}{8\sqrt{2}}.
\]
Therefore
\[ \int_{-\pi}^{\pi} \frac{1}{1 + \sin^2 \theta} d\theta = \frac{4}{i} \times 2\pi i \times \text{(the sum of the residues)} = \frac{4}{i} \times 2\pi i \times \frac{1}{4\sqrt{2}} = \pi \sqrt{2} \]

10. Show that \( \int_{-\infty}^{\infty} \frac{1}{(1 + x^2)^{n+1}} dx = \frac{\pi (2n)!}{2^{2n} (n!)^2} \) for \( n = 0, 1, 2, \ldots \)

Solution: We use an argument similar to that used in question 8. In particular see the remark at the end of that question. The only singularity of \( \frac{1}{(1 + z^2)^{n+1}} \) in the upper half plane is at \( z = i \), and it is a pole of order \( n + 1 \). Therefore
\[
\int_{-\infty}^{\infty} \frac{1}{(1 + x^2)^{n+1}} dx = 2\pi i \text{Residue} \left( \frac{1}{(1 + x^2)^{n+1}}, z = i \right) = \frac{2\pi i}{n!} \frac{d^n}{dz^n} (1 + z^2)^{-n-1} \bigg|_{z=i}
\]
\[
= \frac{2\pi i}{n!} \left( -n-1 \right) (-n-2) \cdots (-n-n) (2i)^{-2n-1} \bigg|_{z=i}
\]
\[
= \frac{2\pi i}{n!} \left( -1 \right) (n+1) (n+2) \cdots (2n) \frac{(2i)^{2n+1}}{(2i)^{2n+1}}
\]
\[
= \frac{\pi}{2^{2n}} \frac{(n+1)(n+2) \cdots (2n)}{n!} \frac{\pi (2n)!}{2^{2n} (n!)^2}
\]