1. Use Cauchy’s Integral Theorem to evaluate the following integrals.

(a) \[ \int_C \frac{z}{z^3 + 1} \, dz, \] where \( C \) is the positively oriented circle \( |z - 2| = 2 \).

(b) \[ \int_C \frac{z}{z^2 + z - 2} \, dz, \] where \( C \) is the circle \( |z| = 3 \), oriented in the clockwise direction.

(c) \[ \int_C \frac{\cos \pi(z - 1)}{z(z^2 + 16)(z^2 - 16)(z^2 + 25)^2(z^2 - 25)^2} \, dz, \] where \( C \) is the circle \( |z| = \pi \) oriented positively.

Solution:

(a) Suppose the partial fraction decomposition of \( \frac{z}{z^3 + 1} \) is

\[ \frac{z}{z^3 + 1} = \frac{A}{z + 1} + \frac{B}{z - e^{\pi i/3}} + \frac{C}{z - e^{-\pi i/3}}. \]

Notice that \( \int_C \frac{A}{z + 1} \, dz = 0 \) since \( z = -1 \) is exterior to \( C \). The other 2 roots are interior to \( C \). Next we compute \( B \) and \( C \):

\[ B = \left. \frac{z(z - e^{-\pi i/3})}{z^3 + 1} \right|_{z = e^{-\pi i/3}} = \frac{e^{-\pi i/3}}{3e^{-2\pi i/3}} = \frac{e^{\pi i/3}}{3} \]

\[ C = \left. \frac{z(z - e^{\pi i/3})}{z^3 + 1} \right|_{z = e^{\pi i/3}} = \frac{e^{\pi i/3}}{3e^{2\pi i/3}} = \frac{e^{-\pi i/3}}{3} \]

Therefore

\[ \int_C \frac{z}{z^3 + 1} \, dz = 2\pi i(B + C) = \frac{2\pi i}{3} \]

(b) The partial fraction decomposition is \( \frac{z}{z^2 + z - 2} = \frac{2/3}{z + 2} + \frac{1/3}{z - 1} \). Since both roots are interior to \( C \) we have

\[ \int_C \frac{z}{z^2 + z - 2} \, dz = 2\pi i \left( \frac{2}{3} + \frac{1}{3} \right) = 2\pi i. \]

(c) The partial fraction decomposition of \( \frac{1}{z(z^2 + 16)(z^2 - 16)(z^2 + 25)^2(z^2 - 25)^2} \) will have the form

\[ \frac{1}{z(z^2 + 16)(z^2 - 16)(z^2 + 25)^2(z^2 - 25)^2} = \frac{A}{z} + \text{other terms}, \]

where the other terms contribute 0 to the integral. Why is this?

Now we compute \( A = \left. \frac{z}{z(z^2 + 16)(z^2 - 16)(z^2 + 25)^2(z^2 - 25)^2} \right|_{z = 0} = -10^{-8} \). Therefore

\[ \int_C \frac{\cos \pi(z - 1)}{z(z^2 + 16)(z^2 - 16)(z^2 + 25)^2(z^2 - 25)^2} \, dz = A \int_C \frac{\cos \pi(z - 1)}{z} \, dz = 2\pi i \times 10^{-8}. \]

Question Where did the minus sign go?
2. Let $C$ be a simple closed contour and let $D$ be its interior. Suppose $f(z)$ and $g(z)$ are analytic in $D$ and on $C$.

(a) Show that $f(z) = g(z) \ \forall z \in D$ if $f(z) = g(z) \ \forall z \in C$.

(b) Show that \[ \int_{C} \frac{f'(\zeta)}{\zeta - z} d\zeta = \int_{C} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \ \forall z \in D. \]

Solution:
(a) By the Cauchy Integral Theorem:
for any $z \in D$, \[ f(z) = \frac{1}{2\pi i} \int_{C} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{C} \frac{g(\zeta)}{\zeta - z} d\zeta = g(z) \]

(b) Apply the Cauchy Integral Theorem twice:
\[ f(z) = \frac{1}{2\pi i} \int_{C} \frac{f(\zeta)}{\zeta - z} d\zeta \implies f'(z) = \frac{1}{2\pi i} \int_{C} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \]
\[ f'(z) = \frac{1}{2\pi i} \int_{C} \frac{f'(\zeta)}{\zeta - z} d\zeta \]

3. Suppose $f(z)$ is an entire function such that $\text{Im}(f(z))$ is bounded. Show that $f(z)$ is a constant.

Solution: Let $f(z) = u(x, y) + iv(x, y)$. Then apply Liouville’s theorem to the entire function $g(z) = e^{f(z)} = e^{-v+iu}$:
\[ |g(z)| = |e^{-v+iu}| = e^{-v} = e^{-\text{Im}(f(z))} \text{ is bounded} \implies f(z) \text{ is a constant}. \]

4. Suppose $P(z) = \prod_{j=1}^{k} (z - r_j)^{s_j}$ is a polynomial in factored form and $C$ is a positively oriented simple closed contour such that $r_1, \ldots, r_n$ are in the interior of $C$ and the rest of the roots are exterior to $C$. Show that \[ \int_{C} \frac{P'(z)}{P(z)} dz = 2\pi i (s_1 + \cdots + s_n). \]

Solution: The partial fraction decomposition of $\frac{P'(z)}{P(z)}$ is $\frac{P'(z)}{P(z)} = \sum_{j=1}^{j=k} \frac{s_j}{z - r_j}$, and therefore
\[ \int_{C} \frac{P'(z)}{P(z)} dz = \sum_{j=1}^{j=k} \int_{C} \frac{s_j}{z - r_j} dz = 2\pi i \sum_{j=1}^{j=n} s_j \text{ since the other roots are exterior to } C \]

5. Evaluate $\int_{C} \frac{e^{az}}{(z^2 + 1)^3} dz$, where $C$ is the circle $x^2 + (y - 1)^2 = 1$, oriented positively.
Solution: The integrand \( \frac{e^{iz}}{(z^2 + 1)^3} \) fails to be analytic only at \( z = \pm i \). First we determine the partial fraction decomposition of \( \frac{1}{(z^2 + 1)^3} \):

\[
\frac{1}{(z^2 + 1)^3} = \frac{A_1}{z - i} + \frac{A_2}{(z - i)^3} + \frac{A_3}{z + i} + \frac{B_1}{(z + i)^2} + \frac{B_2}{(z + i)^3}
\]

\[
A_1 = \frac{1}{2} \frac{d^2}{dz^2} \left( \frac{(z - i)^3}{(z^2 + 1)^3} \right) \bigg|_{z = i} = \frac{1}{2} \frac{d^2}{dz^2} \left( \frac{1}{(z + i)^3} \right) \bigg|_{z = i} = -\frac{3}{16}
\]

\[
A_2 = \frac{d}{dz} \left( \frac{(z - i)^3}{(z^2 + 1)^3} \right) \bigg|_{z = i} = \frac{d}{dz} \left( \frac{1}{(z + i)^3} \right) \bigg|_{z = i} = -\frac{3}{16}
\]

\[
A_3 = \frac{(z - i)^3}{(z^2 + 1)^3} \bigg|_{z = i} = \frac{1}{(z + i)^3} \bigg|_{z = i} = \frac{i}{8}
\]

We don’t need to compute the other numerators since \( \frac{e^{iz}}{(z + i)^n} \) is analytic on and within \( C \) for any integer, and therefore \( \int_C \frac{e^{iz}}{(z + i)^n} \, dz = 0 \). Finally

\[
\int_C \frac{e^{iz}}{(z^2 + 1)^3} \, dz = A_1 \int_C \frac{e^{iz}}{z - i} \, dz + A_2 \int_C \frac{e^{iz}}{(z - i)^2} \, dz + A_3 \int_C \frac{e^{iz}}{(z - i)^3} \, dz
\]

\[
= 2\pi i A_1 e^{iz} \bigg|_{z = i} + 2\pi i A_2 \frac{d}{dz} (e^{iz}) \bigg|_{z = i} + \pi i A_3 \frac{d^2}{dz^2} (e^{iz}) \bigg|_{z = i}
\]

\[
= \frac{3\pi}{8e} + \frac{3\pi}{8e} + \frac{\pi}{8e} = \frac{7\pi}{8e}
\]