1. For each of the following regions $E$, express the triple integral $\iiint_E f(x, y, z) \, dV$ as an iterated integral in cartesian coordinates.

(a) $E$ is the box $[0, 2] \times [-1, 1] \times [3, 5]$;

Solution:

$$\iiint_E f(x, y, z) \, dV = \int_0^2 \int_{-1}^1 \int_3^5 f(x, y, z) \, dz \, dy \, dx$$

(b) $E$ is the pyramid with vertices $(0, 0, 0), (1, 1, 1), (1, 1, -1), (-1, 1, 1)$, and $(-1, 1, -1)$;

Solution:

$$\iiint_E f(x, y, z) \, dV = \int_0^1 \int_{-y}^y \int_{-x}^x f(x, y, z) \, dz \, dy \, dx$$

(c) $E$ is the region in the first octant above the plane $y = z$ and bounded by the cylinder $x^2 + z^2 = 1$.

Solution:

Left function: $y = 0$
Right function: $y = z$

$$\iiint_E f(x, y, z) \, dV = \int_0^1 \int_{\sqrt{1-x^2}}^z f(x, y, z) \, dy \, dx$$
(d) $E$ is the region inside the sphere $x^2 + y^2 + z^2 = 2$ and above the elliptic paraboloid $z = x^2 + y^2$.

**Solution:**

Top function: $z = \sqrt{2 - x^2 - y^2}$

Bottom function: $z = x^2 + y^2$

The boundary of image $D$ on $xy$-plane is the intersection of top and bottom function, which is $x^2 + y^2 = 1$

$$\iiint_E f(x, y, z) \, dV = \int_{-\sqrt{2-x^2-y^2}}^{\sqrt{2-x^2-y^2}} f(x, y, z) \, dz \, dA$$

$$= \int_{-1}^{1} \left( \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y, z) \, dz \right) dy \, dx$$

2. Consider the integral

$$\iiint_E f(x, y, z) \, dV = \int_{-2}^{2} \int_{x^2+y^2}^{4} f(x, y, z) \, dz \, dy \, dx$$

(a) Sketch the region $E$.

**Solution:**

(b) Write the other five iterated integrals which represent $\iiint_E f(x, y, z) \, dV$.

**Solution:**

If we project $E$ onto $xy$-plane, then the top function is $z = y$, and the bottom function is $z = 0$, as given in the question.
In the order of \( dz\,dx\,dy \),

\[
\iiint_E f(x,y,z)\,dV = \int_0^4 \int_{\sqrt{y}}^{\sqrt{y}} \int_0^y f(x,y,z)\,dz\,dx\,dy
\]

If we project \( E \) onto \( yz\)-plane, then the front function is \( x = \sqrt{y} \), and back function is \( x = -\sqrt{y} \).

In the order of \( dx\,dy\,dz \),

\[
\iiint_E f(x,y,z)\,dV = \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_z^y f(x,y,z)\,dx\,dy\,dz
\]

In the order of \( dx\,dz\,dy \),

\[
\iiint_E f(x,y,z)\,dV = \int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_z^y f(x,y,z)\,dx\,dy\,dz
\]

If we project \( E \) onto \( xz\)-plane, we can think of it as a parabolic cylinder \( E_1 : \{(x,y,z) : x^2 < y < 4, -2 < x < 2, 0 < z < 4\} \) (whole body) with the solid \( E_2 : \{(x,y,z) : x^2 < y < z, x^2 < z < 4, -2 < x < 2\} \) (the transparent part) removed. So

\[
\iiint_E f(x,y,z)\,dV = \iiint_{E_1} f(x,y,z)\,dV - \iiint_{E_2} f(x,y,z)\,dV
\]

In the order of \( dy\,dx\,dz \),

\[
\iiint_{E_1} f(x,y,z)\,dV = \int_0^4 \int_{-2}^{2} \int_{x^2}^{y} f(x,y,z)\,dy\,dx\,dz
\]
Figure 6: Q2(b): Left: The image of $E$ on $xz$-plane; Right: solid: $E$, transparent: $E_2$, solid + transparent: $E_1$

\[
\iiint_{E_2} f(x, y, z) \, dV = \int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{x^2}^4 f(x, y, z) \, dy \, dx \, dz
\]

\[
\Rightarrow \iiint_{E} f(x, y, z) \, dV = \int_0^4 \int_{x^2}^4 f(x, y, z) \, dy \, dx \, dz - \int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{x^2}^4 f(x, y, z) \, dy \, dx \, dz
\]

In the order of $dy \, dz \, dx$,

\[
\iiint_{E_1} f(x, y, z) \, dV = \int_{-2}^{2} \int_0^4 \int_{x^2}^4 f(x, y, z) \, dy \, dx \, dz
\]

\[
\Rightarrow \iiint_{E_2} f(x, y, z) \, dV = \int_{-2}^{2} \int_0^4 \int_{x^2}^4 f(x, y, z) \, dy \, dx \, dz
\]

\[
\Rightarrow \iiint_{E_2} f(x, y, z) \, dV = \int_{-2}^{2} \int_0^4 \int_{x^2}^4 f(x, y, z) \, dy \, dx \, dz - \int_{-2}^{2} \int_0^4 \int_{x^2}^4 f(x, y, z) \, dy \, dx \, dz
\]

(c) Find the volume of $E$.

**Solution:**

The volume of $E$ is given by

\[
\iiint_{E} dV = \int_{-2}^{2} \int_0^4 \int_0^y dz \, dy \, dx = \int_{-2}^{2} \int_0^4 y \, dy \, dx = \int_{-2}^{2} \left[ 8 - \frac{1}{2} x^4 \right] \, dx = \frac{128}{5}
\]

(d) Find the centre of mass of $E$ when the density of $E$ is constant.

**Solution:**

Let the constant density be $\rho(x, y, z) = c$. Then

\[
m = \iiint_{E} c \, dV = c(\text{volume of } E) = \frac{128c}{5}
\]

\[
\bar{x} = \frac{1}{m} \iiint_{E} cx \, dV = \frac{1}{m} \int_{-2}^{2} \int_0^4 \int_0^y cx \, dy \, dz \, dx = \frac{5}{128c}(0) = 0
\]

\[
\bar{y} = \frac{1}{m} \iiint_{E} cy \, dV = \frac{1}{m} \int_{-2}^{2} \int_0^4 \int_0^y cy \, dy \, dz \, dx = \frac{5}{128c} \frac{512}{7} = \frac{20}{7}
\]

\[
\bar{z} = \frac{1}{m} \iiint_{E} cz \, dV = \frac{1}{m} \int_{-2}^{2} \int_0^4 \int_0^y cz \, dy \, dz \, dx = \frac{5}{128c} \frac{256}{7} = \frac{10}{7}
\]

The centre of mass is \( (0, \frac{20}{7}, \frac{10}{7}) \).
3. Let $E$ be the solid bounded by $z = \sqrt{x^2 + y^2}$ and $z = \sqrt{1 - x^2 - y^2}$.

(a) Use cylindrical coordinates to find the volume of $E$.

Solution:

$$V = \int \int \int_E dV = \int \int_D \int_{\sqrt{1-r^2}}dz \, dA$$

$D$ is the circular image on $xy$-plane. The boundary of $D$ is given by the intersection of the top and bottom function, which is $x^2 + y^2 = 1/2$, or $r = \sqrt{1/2}$.

$$V = \int_0^{\pi/2} \int_0^{\sqrt{1/2}} \int_{\sqrt{1-r^2}} r \, dz \, dr \, d\theta = \frac{\pi}{3} (2 - \sqrt{2})$$

(b) Use spherical coordinates to find the volume of $E$.

Solution:

Write the function in spherical coordinates: $z = \sqrt{x^2 + y^2} \Rightarrow \phi = \pi/4$; $z = \sqrt{1 - x^2 - y^2} \Rightarrow \rho = 1$. So

$$V = \int \int \int_E dV = \int_\phi^{\pi/4} \int_0^{2\pi} \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{\pi}{3} (2 - \sqrt{2})$$

4. Find the volume of the solid above the $xy$-plane, under the surface $z = 1 - x^2 - y^2$, and within the wedge $x \leq y \leq \sqrt{3}x$.

Solution:

Top function: $z = 1 - x^2 - y^2$

Bottom function: $z = 0$

$$V = \int \int \int_E dV = \int \int_D \int_0^{1-x^2-y^2} dz \, dA$$

Since on the $xy$-plane, $0 = 1 - x^2 - y^2 \Rightarrow r = 1$, $y = x \Rightarrow \theta = \pi/4$ and $y = \sqrt{3}x \Rightarrow \theta = \pi/3$, $D$ is the region on the $xy$-plane defined by $D : \{ (r, \theta) : 0 < r < 1, \pi/4 < \theta < \pi/3 \}$.

So

$$V = \int_\phi^{\pi/4} \int_0^1 \int_{1-r^2}^{1-x^2-y^2} r \, dz \, dr \, d\theta = \frac{\pi}{48}$$
5. Find the volume remaining in a sphere of radius $a$ after a hole of radius $b$ is drilled through the centre. Assume $0 < b < a$.

**Solution:**

Sphere of radius $a$: $x^2 + y^2 + z^2 = a^2$

Top function: $z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - r^2}$

Bottom function: $z = -\sqrt{a^2 - x^2 - y^2} = -\sqrt{a^2 - r^2}$

Cylindrical hole of radius $b$: $x^2 + y^2 = b^2 \Rightarrow r = b$

$$V = \iiint\limits_E dV = \int_0^{2\pi} \int_b^a \int_{\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r \, dz \, dr \, d\theta = \frac{4}{3} \pi (a^2 - b^2)^{3/2}$$
Solution:

Outer function: \( x^2 + y^2 + z^2 = 4 \Rightarrow \rho = 2 \)

Inner function: \( x^2 + y^2 + z^2 = 1 \Rightarrow \rho = 1 \)

\( xy \)-plane: \( \phi = \pi/2 \)

\[
m = \iiint_E z \ dV = \int_0^{\pi/2} \int_0^{2\pi} \int_1^2 (\rho \cos \phi) \rho^2 \sin \phi \ d\rho \ d\theta \ d\phi
\]
\[
= \int_0^{\pi/2} \int_0^{2\pi} \int_1^2 \rho^3 \cos \phi \sin \phi \ d\rho \ d\theta \ d\phi
\]
\[
= \frac{15}{4} \pi
\]