

(1) A spring has a natural length of 16 cm. When it is stretched  $x$  cm beyond that, Hooke's Law states that the spring pulls back with a restoring force  $F = kx$  dynes, where the constant  $k$  is called the spring constant, and represents the stiffness of the spring. For the given spring, 8 dynes of force are required to hold it stretched by 2 cm. How much work (dynes-cm) is done in stretching this spring from its natural length to a length 24 cm?

**Solution:** Since  $F = 8$  when  $x = 2$  we find that  $8 = 2k$  so that  $k = 4$  and  $F = 4x$  is the force. When the spring is at its natural length, no work is done. To stretch the spring by  $\Delta x$  beyond this would require work  $W = F\Delta x = 4x\Delta x$ . To sum up all the work involved in stretching from 16 to 24 cm, i.e. for the increase in length  $x = 0$  to  $x = 8$ , we would calculate  $W = \int F dx = \int_0^8 4x dx = 2x^2|_0^8 = 128$  ergs.

(2) Calculate the work done in pumping water out of a parabolic container. Assume that the container is a surface of revolution generated by rotating the curve  $y = x^2$  about the  $y$  axis, that the height of the water in the container is 10 units, that the density of water is  $1gm/cm^3$  and that the force due to gravity is  $F = mg$  where  $m$  is mass and  $g = 9.8m/s^2$ .

**Solution:** The paraboloid can be thought of as a bowl full of water. The total work done is equivalent to the sum of work done to pump each bit of the water up to the height 10, or equivalently, a sum over "disks" of water, each at some height  $y$ . The work done is **Work = (force) (distance)** and the Force is  $F = mg$  where  $g = 9.8m/s^2 = 980cm/s^2$ . The mass of a disk at height  $y$  is  $m_y = (1gm/cm^3)(\pi r^2 \Delta y \text{ cm}^3)$  where the radius of the disk is the  $x$  coordinate of a point on the parabola  $y = x^2$ . Thus, the work done to move this disk to height 10 cm is  $w_y = m_y g(10 - y) = 980\pi(10 - y)y\Delta y$ , and the total work, which is a sum over all disks is  $W = 980\pi \int (10 - y)y dy = 980\pi \int (10y - y^2) dy = 980\pi [10(y^2/2) - (y^3/3)]|_0^{10} = 980\pi 10^3 [(1/2) - (1/3)] = 5.128 \cdot 10^5$  ergs. (One erg is  $1 gm \text{ cm}^2 s^{-2}$  or 1 dyne-cm).

(3) In a harmonic oscillator (such as a spring) the energy is partly kinetic (associated with motion), and partly potential (stored in the stretch of the spring). Energy conservation implies that the sum of the two forms ( $E$ ) is constant so that

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E$$

where  $m$  is the mass on the spring,  $k$  the spring constant.

(a) Use the fact that the velocity  $v = dx/dt$  to show that this energy conservation equation

implies that

$$\frac{dx}{dt} = \left(\frac{2E - kx^2}{2m}\right)^{1/2}$$

- (b) Rewrite this in integral form by dividing both sides of the equation by the term in the square-root (this is called "separation of variables") to obtain

$$\int \frac{dx}{\left(\frac{2E - kx^2}{2m}\right)^{1/2}} = \int dt$$

- (c) Integrate both sides. You will need to use substitution, and the result will be an inverse trig function for  $x$ .
- (d) Find the displacement  $x$  as a function of time  $t$ . Show that the motion of the spring is periodic.

**Solution:** (a) The velocity  $v = dx/dt$  so that  $\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 = E$ , so that  $m\left(\frac{dx}{dt}\right)^2 = 2E - kx^2$ . Simplifying gives  $\frac{dx}{dt} = \left(\frac{2E - kx^2}{m}\right)^{1/2}$

(b)  $\int \frac{dx}{\left(\frac{2E - kx^2}{m}\right)^{1/2}} = \int dt$  is obtained by "cross multiplying" and integrating both sides, and is called separation of variables.

(c) We simplify to  $\int \frac{dx}{\sqrt{\frac{k}{m}\sqrt{(2E/k) - x^2}}} = t + C$ . Let  $a^2 = 2E/k$ . Then the above is  $\sqrt{\frac{m}{k}} \int \frac{dx}{\sqrt{a^2 - x^2}} = t + C$ . The integral on the left is a trig-type integral, and we get  $\sqrt{\frac{m}{k}} \sin^{-1}(x/a) = t + C$ , so  $\sin^{-1}(x/a) = \sqrt{\frac{k}{m}}(t + C)$ , i.e.  $(x/a) = \sin[\sqrt{\frac{k}{m}}(t + C)]$ . Thus  $x = a \sin[\sqrt{\frac{k}{m}}(t + C)] = \sqrt{\frac{2E}{k}} \sin[\sqrt{\frac{k}{m}}(t + C)]$ .

(d) This is a periodic function with frequency  $\omega = \sqrt{\frac{k}{m}}$  and amplitude  $\sqrt{\frac{2E}{k}}$ .

**(4)** (a) The temperature of a cup of juice is observed to be  $T(t) = 25(1 - e^{-0.1t})$  degrees Celsius where  $t$  is time in minutes. Find the change in the temperature of the juice between  $t = 1$  and  $t = 5$ . (b) The **rate of change** of temperature of a cup of coffee is observed to be  $f(t) = 80e^{-0.2t}$  degree Celsius per minute where  $t$  is time in minutes. What is the **total change** in the temperature between  $t = 1$  and  $t = 5$  minutes?

**Solution:** We have in (a)

$$T(t) = 25 - 25e^{-0.1t}$$

The temperature starts at 0 and rises to 25 as  $t$  gets large. The answer is

$$T(5) - T(1) = 25e^{-0.1} - 25e^{-0.5} = 7.46$$

(5) The density of a band of protein along a one-dimensional strip of gel in an electrophoresis experiment is given by  $p(x) = 2(x - 1)(2 - x)$ , for  $1 \leq x \leq 2$  where  $x$  is in cm and  $p$  is in protein mass per cm. (a) Graph the density  $p$  as a function of  $x$ . (b) Find the total mass of the protein in the band for  $1 < x < 2$ . [Hint: simplify the function first.]

**Solution:** The graph is a hump between 1 and 2.

The total mass is the integral of density, or

$$\begin{aligned} \int_1^2 2(x - 1)(2 - x) dx &= \int_1^2 (2x - 2)(2 - x) dx \\ &= \int_1^2 -2x^2 + 6x - 4 dx \\ &= [-2x^3/3 + 3x^2 - 4x]_1^2 \\ &= 1/3 \end{aligned}$$

(6) The density of bacteria growing in a circular colony of radius 1 cm is observed to be  $b(r) = 1 - r^2$ ,  $0 \leq r \leq 1$  (in units of one million cells per square centimeter), where  $r$  is the distance (in cm) from the center of the colony. (a) Graph the density as a function of distance from the center of the colony. (b) What is the total number of bacteria in the colony?

**Solution:** Group the bacteria in bands at a fixed radius, thickness  $dr$ . The area of the band is  $2\pi r dr$ . Since the density is  $1 - r^2$ , it is constant in the band, and the number is  $2\pi r(1 - r^2)$ . The total number is (in millions)

$$\int_0^1 2\pi r(1 - r^2) dr = [\pi r^2 - \pi r^4/2]_0^1 = \pi/2$$

(7) A test-tube of height 10 cm and radius 2 cm contains a **density gradient** of glucose (i.e. the concentration varies from the top to the bottom). The concentration of glucose as a function of the depth,  $x$  is  $c(x) = (x/10)$  (in units of gm/cm<sup>3</sup>). (Note:  $x = 0$  at the top of the tube). What is the total amount of glucose in the tube (in gm)?

**Solution:** The total amount is

$$\int_0^{10} (x/10)(\pi 2^2) dx = (2\pi/5)(10^2/2) .$$

(8) Normal exposure to radiation from natural sources is approximately 0.1 rad per year. But for people living next to a site such as the Hanford nuclear plant where radioactive

nuclear waste was released into the environment, exposure can be much higher. Damage due to radiation is cumulative. Suppose the radio isotope buried in the drums under your backyard has a half-life of one month. Initial exposure to inhabitants in your house when you moved in was 0.5 rad per month. (A rad is a unit of basic radioactive material.) Calculate the cumulative exposure in rads that would occur if you lived there for 10 years. (Recall that radioactive materials decay exponentially, and that radioactivity is proportional to the amount of radioactive material.)

**Solution:** A half-life of one month means the amount of radioactivity  $r(t)$  is

$$r(t) = r(0)2^{-t} = 0.52^{-t} = 0.5e^{-t \ln(2)} \text{ rads/month}$$

where  $t$  is measured in months. This is the rate at which radioactive material is let off. The total let off is the anti-derivative of this, or

$$R(t) = \frac{-0.5e^{-t \ln(2)}}{\ln(2)} = \frac{-0.52^{-t}}{\ln(2)}$$

and in 10 years amounts to

$$R(10) - R(0) = \frac{0.5}{\ln(2)}(1 - 2^{-10}) .$$

(9) The growth rate of a crop is known to depend on temperature during the growing season. Suppose growth rate of the crop in tons per day is given by  $g(t) = 0.1(T - 18)$  where  $T$  is temperature in degrees Celsius. Suppose the temperature record during the 90 days of the season was  $T(t) = 22 + (0.1)t + 4 \sin(2\pi t/60)$  where  $t$  is time in days. Find the total growth (in tons) that would have occurred.

**Solution:** The growth rate is

$$g(t) = 0.1(4 + (0.1)t + 4 \sin(2\pi t/60))$$

and the total grown up to time  $T$  is

$$\begin{aligned} G(T) &= \int_0^T 0.1(4 + (0.1)t + 4 \sin(2\pi t/60)) dt \\ &= 0.1 \int_0^T 4 + (0.1)t + 4 \sin(2\pi t/60) dt \\ &= 0.1 [4t + (0.05)t^2 - 4 \cos(2\pi t/60)]_0^T \\ G(90) &= 0.1(360 + 405 - (120/\pi)(\cos 3\pi - \cos 0)) \\ &= 84.14 \text{ tons} \end{aligned}$$

(10) The infant mortality rate (in deaths per year per capita) for two developing nations is shown in this figure:

Picture missing.

Suppose that the two nations have the same constant population of infants between age 0 and 11 year old. (a) Sketch the total expected mortality of infants over a period of 1 year. (b) Which of the two populations has a greater total mortality for infants between age 0 and 0.5 years? Between age 0.5 and 1 year? Between age 0 and 1 year?

**Solution:** Solution missing.

(11) Oil leaks out of an oil tanker at the rate  $f(t) = 10 - (0.2)t^2$  (where  $f$  is in units of 10,000 barrels per hour and  $t$  is in hours). (Note: This function only makes sense as long as  $f(t) \geq 0$  since a negative flow of oil is meaningless in this case.) (a) At what time will the flow be zero? (b) What is the total amount that has leaked out between  $t = 0$  and this time?

**Solution:** The flow will be 0 when  $10 = (0.2)t^2$  or  $t = \sqrt{50}$ . The total by then will be

$$\int_0^{\sqrt{50}} 10 - (0.2)t^2 dt = [10t - (0.2)t^3/3]_0^{\sqrt{50}} = 47.14 \text{ (ten thousand barrels)}$$

(12) After World War II, the birth rate increased dramatically. Suppose that the number of babies born (in millions per year) was given by  $b(t) = 10 + (0.01)t^3$ ,  $0 < t < 10$  where  $t$  is time in years after the end of the war. (a) How many babies in total were born during this time period? (b) At what time  $t$  was the total number born precisely 14 million?

**Solution:** The total at time  $t$  was

$$\int_0^t 10 + (0.01)s^3 ds = [10s + (0.01)s^4/4]_0^t = 10t + (0.01)t^4/4.$$

At  $t = 10$  this is 125 (million). For (b) we set this equal to 14 and solve for  $t$

$$10t + (0.01)t^4/4 = 14$$

We can easily see that there is a solution between 1 and 2. Trial and error give us  $t = 1.399$ .

(13) Let

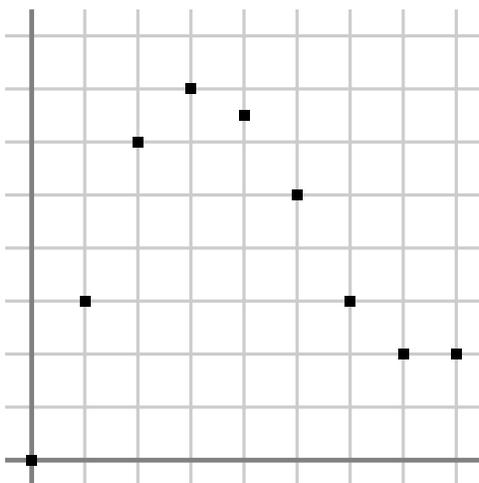
$$g(t) = \int_0^t f(s) ds$$

where  $f(t)$  is the function whose graph is shown ( $t \geq 0$ ). The grid in this figure is  $1 \times 1$ , and the axes are darkened.



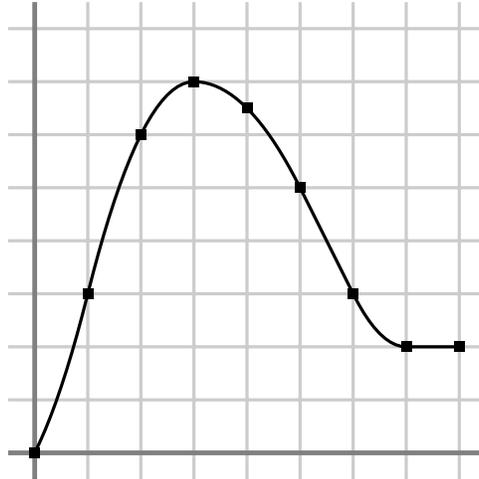
Draw the graph of  $g(t)$  as accurately as you can.

**Solution:** We can plot points on the graph with integral values of  $x$  just by counting squares:



But we cannot just connect these points by straight lines to produce the graph of  $g(t)$ . For example, between  $t = 0$  and  $t = 1$  we have  $f(t) = 2 + 2t$ , so that in fact  $g(t)$  must be its anti-derivative  $2t + t^2$  in the same range. We can sketch the graph by a simpler procedure, however; the derivative of  $g(t)$  at the various points must be equal to  $f(t)$ . For example,  $g'(0) = 2$  but  $g'(1) = 4$ ,  $g'(2) = 2$ ,  $g'(3) = 0$ , etc. So we must not just connect

the dots, we must connect dots and slopes. When we take all these things into account, we get this graph:



(14) An express mail truck delivers mail to various companies situated along a central avenue and often goes back and forth as new mail arrives. Over some period of time,  $0 \leq t \leq 10$ , its velocity (in km per hour) can be described by the function  $v(t) = t^2 - 9t + 14$ . Find:

(a) The net distance (i.e. displacement) traveled over this period of time. [Hint: recall that if you leave home in the morning, travel to UBC, and then go back home, then your net distance traveled, or total displacement over this full period of time is zero.]

(b) How much gasoline was consumed during this period of time if the vehicle uses 5 liters per km. [Hint: now you will need to find the total distance that the vehicle actually covered during its trip.]

**Solution:** The formula for net distance is

$$\int_0^t s^2 - 9s + 14 ds = [s^3/3 - 9s^2/2 + 14s]_0^t$$

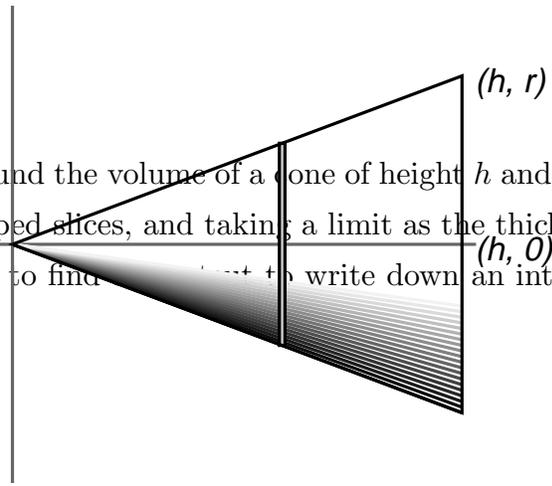
When  $t = 10$  this is 23.33 km.

The velocity  $(t - 7)(t - 2)$  is positive to start with, but reverses at  $t = 2$  and again at  $t = 7$ . So the total distance traveled is

$$[s^3/3 - 9s^2/2 + 14s]_0^2 - [s^3/3 - 9s^2/2 + 14s]_2^7 + [s^3/3 - 9s^2/2 + 14s]_7^{10} = 65$$

This makes 325 litres of gas used.

(15) In a previous homework problem, we found the volume of a cone of height  $h$  and base radius  $r$  by summing up volumes of disk-shaped slices, and taking a limit as the thickness of the slices went to 0. Here we use calculus to find the volume. Write down an integral representing this volume.



We will place the cone on its side, as shown in the figure, and let  $x$  represent position along its axis. The straight line shown will generate the surface of the cone by revolving about the  $x$  axis. (a) What is the equation of this line? (b) Slice the cone perpendicularly to the  $x$ -axis by slices distributed along the  $x$  axis (i.e. into vertical disks) with very small slice thickness  $\Delta x$ . What is the radius, and hence the volume of a slice at position  $x$ ? (c) Write down the integral which calculates the volume of the cone and find this volume.

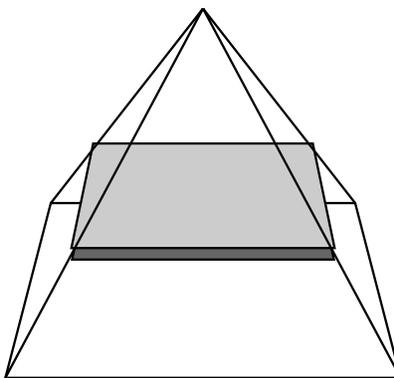
**Solution:** (a)  $y = (r/h)x$ .

(b) Radius is  $(r/h)x$ , volume is  $\pi(r^2x^2/h^2) \Delta x$ .

(c) It is

$$\int_0^h \pi r^2 x^2 / h^2 dx = \pi r^2 h / 3 .$$

(16) In this problem you will find the volume of a pyramid with a square base of width  $w$  and with a height  $h$ .



Let the variable  $z$  stand for distance **down** the axis of the pyramid with  $z = 0$  at the top, and consider slicing the pyramid perpendicularly to this axis (thus into horizontal slices).

This will produce square slices (having area  $A(z)$  and some thickness  $dz$ ). Calculate the volume of the pyramid as an integral by figuring out how  $A(z)$  depends on  $z$  and integrating it.

**Solution:** Let the side length of the square cross-section at height  $z$  be  $w(z)$ . Then by similar triangles,  $w(z)/z = w/h$ , or  $w(z) = (w/h)z$ . Now the area of the square slice at height  $z$  is  $A(z) = w(z)^2 = (w/h)^2 z^2$  and its thickness is  $dz$ . Thus

$$V = \int_0^h (w/h)^2 z^2 dz = (w/h)^2 [z^3/3]_0^h = w^2 h/3$$

(17) The trunk of a tree grows incrementally, so that its cross-section (as is well known) consists of tree-rings. Suppose that over a period of 10 years, the growth rate of the radius of the tree trunk (in cm/year) is given by the function  $f(t) = 1.5 + \sin(\pi t/5)$ . (a) If the radius was initially  $r_0$  at time  $t = 0$ , what will the radius of the trunk be  $t$  years later? Suppose the height of the tree (and its trunk) is approximately constant over this ten year period. (b) What will be the ratio of its final and initial mass? (c) Suppose that  $r_0 = 1$ . Over which year will the net increase of the mass of the tree trunk be largest? Same question with  $r_0 = 100$ .

**Solution:** (a) The radius at time  $t$  is

$$r_0 + \int_0^t 1.5 + \sin(\pi s/5) ds = r_0 + 1.5t - \frac{5}{\pi} [\cos(\pi s/5)]_0^t = r_0 + 1.5t + \frac{5}{\pi} (1 - \cos(\pi t/5))$$

(b) The ratio of volumes is equal to the ratio of sectional areas. We have

$$A(t) = \pi r(t)^2 = \pi r_0^2 \left( 1 + \frac{1.5t}{r_0} + \frac{5}{\pi r_0} (1 - \cos(\pi t/5)) \right)^2$$

Set  $t = 10$  and calculate.

(c) For this, we have to make tables. First with  $r_0 = 1$ .

$t$	0	1	2	3	4	5	6	7	8	9	10
radius	1.00	2.80	5.10	7.58	9.88	11.68	12.88	13.58	14.10	14.80	16.00
area	3.14	24.70	81.70	180.66	306.61	428.81	521.10	579.65	624.56	688.50	804.25
difference	21.56	57.00	98.96	125.95	122.20	92.29	58.55	44.91	63.95	115.75	

So the maximum growth occurs during the fourth year. With  $r_0 = 100$  it occurs during the third year. The rough idea is that the amount of cross-sectional area added each year depends on both the radius and the growth rate of the radius. Even if the growth of the radius is small, if the radius itself is large then the amount of area added will be large. This factor plays a more important role when the initial radius is small. (Why is that?)