**QUESTION 1:**

Find the derivatives of the following functions. DO NOT TRY TO SIMPLIFY.

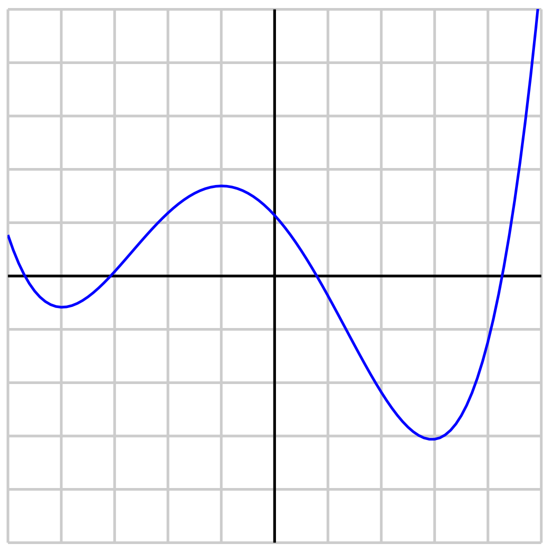
(a) $f(x) = \tan(1/x)$ (b) $f(x) = \sin(e^x)$ (c) $f(x) = (1 + \cos(x))^{1/3}$ (d) $f(x) = \dfrac{\ln x}{x+1}$

**QUESTION 2:**

Using only the definition of the derivative find $f'(x)$ for the function $f(x) = \sqrt{x^2+1}$.

**QUESTION 3:**

Shown below is the graph of a function $y = f(x)$.



Sketch the graph of the function $f'(x)$.

**QUESTION 4:**

Suppose that a bird foraging for a time $t$ derives, from the food gathered, an amount of energy equal to $G(t) = \dfrac{3t}{1+t}$. At the same time, because of its foraging effort, it loses an amount of energy equal to $L(t) = 2t$.

(a) What is the net gain in energy, $E(t)$, after foraging for a time $t$?

(b) For which values of $t > 0$ (that is, for which amounts of time spent foraging) is the net gain in energy zero?

(c) Determine that time $t$ for which the net gain in energy is greatest.

**QUESTION 5:**

Let $f(x) = \dfrac{x}{2} - \sin x$, $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

(a) Find all critical ponts of $f(x)$ and determine if they are local maxima, local minima or neither.

(b) Find all intervals where $f(x)$ is increasing (resp. decreasing).

(c) Find all intervals where $f(x)$ is concave up (resp. concave down).

**QUESTION 6:** A certain function $f(x)$ satisfies $f(1) = 2,\ f'(1) = -1$.

(a) What is the equation of the tangent line to the graph of $y = f(x)$ when $x = 1$?

(b) Find an approximation for $f(1.01)$.

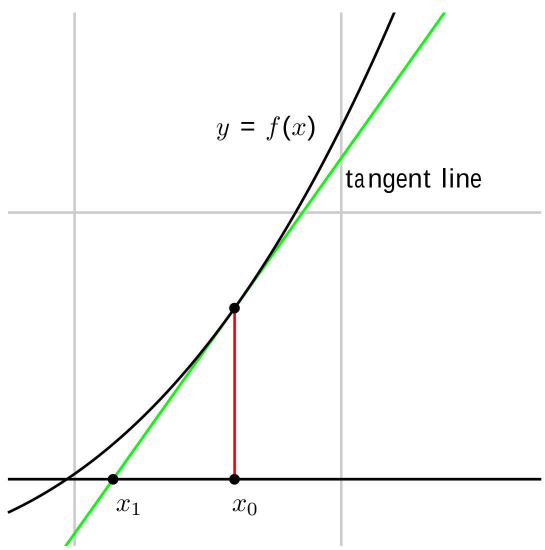
Now assume in addition that $f''(x) > 0$ for all $x$.

(c) Sketch the graph of $y = f(x)$ in the neighbourhood of $x = 1$.

(d) Will your approximation in (b) be larger or smaller than the actual value of $f(1.01)$?

**QUESTION 7:**

(a) Derive a formula for $x_1$ in terms of $x_0$. See the diagram below.




(b) Use Newton's method to find the critical point(s) of the function $g(x) = x^2 + e^{-x}$.

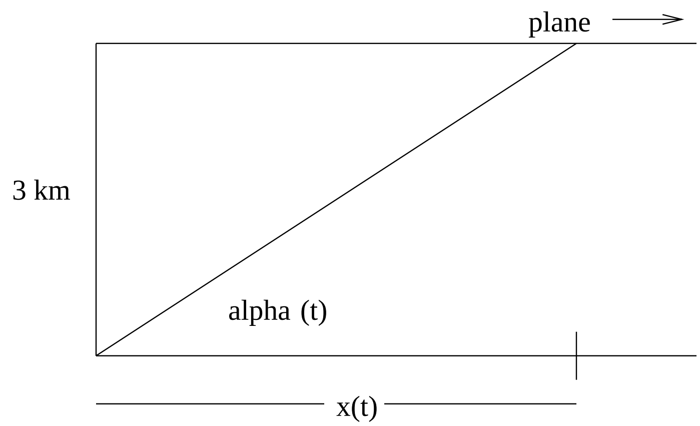
**QUESTION 8:**

An airplane travelling at $400km/hr$ and at a constant altitude of $3km$ passes directly over a radar station on the ground which is tracking the plane.

(a) How far does the plane fly in $t$ hours?

(b) Determine the angle of elevation $\alpha(t)$, see the diagram.

(c) Determine the rate of change of the angle of elevation $20sec$ after the plane passes overhead.




**QUESTION 9:**

Cultures of two different cell types, A and B, are studied in a biotech lab. Let $y_A(t)$ and $y_B(t)$ be the concentrations of the two at time $t$. Initially, the two concentrations are identical, i.e. $y_A(0) = y_B(0) = y_0$. After 1 day, both $y_A$ and $y_B$ doubled, i.e. $y(1)_A = y(1)_B = 2y_0$. During the 2nd day, $y_A$ doubled again, i.e. $y_A(2) = 4y_0$, but $y_B(2) = y_B(1) + y_0 = 3y_0$. This trend continued, i.e. $y_A$ doubled every day while $y_B$ increased by $y_0$ each day.

(a) Find the two concentrations at any time $t$ (in days), that is $y_A(t)$ and $y_B(t)$.

(b) How long does it take for each population to reach 100 times the initial concentration?

**QUESTION 10:**

A room containing 1200 ft$^3$ of air is orginally free of carbon monoxide. At time $t = 0$ cigarette smoke that contains 4% of carbon monoxide is produced at the rate of 0.1 ft$^3$/min by a couple of smokers. The well-circulated mixture is allowed to leave the room at the same rate.

(a) Find an expression for the concentration $x(t)$ (percentage per ft$^3$) of carbon monoxide in the room at any time $t$ after smoking started.

(b) A carbon monoxide concentration as low 0.012% per ft$^3$ is harmful at extended exposure. How long does it take for $x(t)$ to reach this level?

**QUESTION 11:**

Use Newton's method to find an approximate value for $\sqrt{8}$.

**QUESTION 12:**

Use Newton's method to find an approximation (correct to $\pm 0.01$ ) for any roots of the equation

$$\sin(x) = \frac{1}{2}x$$

How many roots does this equation actually have ? Draw a diagram showing the functions $y = \sin(x)$ and $y = x/2$ on the same set of axes to help answer this question.

**QUESTION 13:**

Use Newton's method to find critical points of the functions:

(a) $y = x^4 + x^3 - 4x^2 - x + 1$.

(b) $y = e^x - 2x^2$

**QUESTION 14:**

For each of the following equations, determine the slope of the tangent line $dy/dx$ at the indicated point:

(a) $x^3 + yx^2 + y^2x + y^3 = 1$ at the point $(1, 1)$.

(b) $\sin(x) + x\cos(y) = y$ at the point (0,0).

(c) $e^{x^2+y^2} = 10xy$ at the point $(x, y)$.

**QUESTION 15:**

A ladder of length $L$ is leaning against a wall so that its point of contact with the ground is a distance $x$ from the wall, and its point of contact with the wall is at height $y$. The ladder slips away from the wall at a constant rate $C$.

(a) Find the rate of change of the height $y$.

(b) Find the rate of change of the angle $\theta$ formed between the ladder and the wall.

**QUESTION 16:**

A car is being pulled out of mud by a tow truck. The pulley on the truck is 1 m higher than the bumper on the car. If the rope is being pulled in at the rate of 0.5m/min, how quickly is the angle between the rope and the horizon changing when the car is 5 m away from the tow truck?.

**QUESTION 17:**

Two sides of a triangle are made of elastic bands that are being extended at constant rates $k_1, k_2$ respectively. The third side is of fixed length. At what rate is the angle between the two elastic bands changing at the instant when they have lengths $l_1$ and $l_2$ ? Hint: Use the Law of Cosines which relates the third side of any triangle to the known lengths of two sides and the angle between them:

$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$

**QUESTION 18:**

Jack and Jill have an on-again off-again love affair. Their love (or dislike) for one another at time $t$ is given by the function $y(t) = \sin(2t) + \cos(2t)$.

(a) Find the times when their love is at a maximum.

(b) Find the times when they dislike each other the most.

**QUESTION 19:**

A ship sails away from a harbour at a constant speed. The total height of the ship including its mast is $h$.

(a) At what distance away will the ship disappear below the horizon?

(b) At what rate does the top of the mast appear to drop towards the horizon just before this? (Note: In ancient times this effect lead people to conjecture that the earth is round, a fact which you need to take into account in solving the problem.)

**QUESTION 20:**

Two populations are studied. Population **1** obeys the differential equation $dy_1/dt = ky_1$, whereas population **2** obeys $dy_2/dt = -ky_2$, where $k$ is a positive constant.

(a) Which population is growing and which is declining?

(b) If the initial levels of the two populations were identical, say $y_1(0) = y_2(0) = 1000$, how big would each population be at some time $t$ later?

(c) Determine the half-life (time at which only half of the population is left in the case of the decreasing population) and the doubling time (time at which the population has doubled in the case of the growing population) for these cases.

**QUESTION 21:**

(a) The population of a certain microorganism grows continuously and follows an exponential behaviour over time. Its doubling time is found to be 0.27 hours. What differential equation would you use to describe its growth ? (Note: you will have to find the value of the rate constant, $k$ using the doubling time.)

(b) With exposure to ultra-violet radiation, the population ceases to grow, and the microorganisms continuously die off. It is found that the half-life is then 0.1 hours. What differential equation would now describe the population?

**QUESTION 22:**

A culture of bacteria is found to contain $10^4$ bacteria per $cm^3$ at the start of an experiment. After 1 day there are $10^6$ bacteria per $cm^3$. Assume that the number of bacteria increases at a rate proportional to the number of bacteria present. Determine the doubling time of the bacteria. How many will there be after 2 days?

**QUESTION 23:**

(a) The half-life of radium is 1620 years. What percentage of a sample of radium will remain after 500 years?

(b) Cobalt 60 is a radioactive substance with half life 5.3 years. It is used in medical applications (radiology). How long does it take for 80% of a sample of this substance to decay?

**QUESTION 24:**

In a chemical reaction it is found that a substance B is broken down at a rate proportional to the amount of substance remaining. It was observed that 10 g of B decreased to 8 gm in 1 hr. How long will it take until only 1 gm is left?

**QUESTION 25:**
Sketch the following functions:

$$\text{(a) } y = \frac{e^x - e^{-x}}{2} \qquad\qquad \text{(b) } y = x^2 e^x$$

Identify any critical points, and determine whether these are local maxima or minima.

**QUESTION 26:**

The time of death of a murder victim can be estimated from the temperature of the body if it is discovered early enough after the crime has occurred. Suppose that in a room whose ambient temperature is $E = 20$ degrees centigrade, the temperature of the body upon discovery is $T = 30$ degrees, and that a second measurement, one hour later is $T = 25$ degrees. Determine the approximate time of death. (You should use the fact that just prior to death, the temperature of the victim was 37 degrees).

**QUESTION 27:**

Newton's second law (of mechanics) states that force is proportional to acceleration ($F = ma$, where $m$ is mass). For an object falling due to gravity and experiencing a frictional force, this law leads to

$$ma = mg - kv$$

where $g, k$ are positive constants and $v$ is velocity.

(a) Use the fact that $a = dv/dt$ to express this relationship in the form of a differential equation for the velocity $v$.

(b) Show that the function $v(t) = \dfrac{mg}{k}(1 - e^{-kt/m}) + v_0 e^{-kt/m}$ satisfies this differential equation. Here $v_0$ is the initial velocity.

(c) What happens to the velocity after a long time, assuming the object is still falling?

**QUESTION 28:**
A tank that holds 1000 litres is initially full of water to which 100 kg of salt has been added. (The mixture was stirred so that the salt concentration was then 0.1 kg/litre). A concentrated solution of salt, containing 0.25 kg/litre is pumped into the tank continuously, at the rate 10 litres/min and the mixture (which is continuously stirred to keep it uniform)

is pumped out at the same rate. How much salt will there be in the tank after 10 minutes? After 1 hr?

**QUESTION 29:** For what values of $a$ does the curve $y = a^x$ intersect the curve $y = x$?

**QUESTION 30:**

Two cars are approaching an intersection. The car travelling due south is travelling at velocity 50 km/hr. The car moving due east is speeding at 80 km/hr. At what rate is the distance between the cars changing when they are each 1 km away from the intersection? (Assume that the roads are perpendicular.)

**QUESTION 31:**

Sketch the function

$$y = f(x) = 2e^{-x^2} - e^{-x^2/3}.$$

Comment on why it might be called "a Mexican Hat" function. Find any local minima and maxima.

**QUESTION 32:**

The distribution of seeds at a distance $x$ from a parent tree is often found to be

$$D(x) = D_0 e^{-x^2/a^2},$$

where $a > 0, D_0 > 0$ are positive constants. Insects that eat these seeds tend to congregate near the tree so that the fraction of seeds that get eaten may be

$$F(x) = e^{-x^2/b^2},$$

where $b > 0$. (Remark: These functions are called Gaussian or Normal distributions. The parameters $a, b$ are related to the "width" of these bell-shaped curves.) The number of seeds that survive (i.e. are produced and not eaten by insects) is $S(x) = D(x)(1 - F(x))$. Assume that $b < a$. Determine the distance $x$ from the tree at which the greatest number of seeds survive.

**QUESTION 33:**

Use implicit differentiation to find points on the ellipse $x^2 - 4y^2 = 4$ at which the slope of the tangent line is 0.5.

**QUESTION 34:**
Find all the critical points of the function given below and determine what kind of critical point each one is. (Your answer should be given in terms of the constant $a$, and you may assume that $a > 0$.)

$$y = f(x) = 2x^3 + 3ax^2 - 12a^2x + 1$$

**QUESTION 35:**

Find two numbers whose sum is 20 and whose product is as large as possible.

**QUESTION 36:**

A ball is thrown from the top of a building whose height is $h_0$, with initial velocity $v_0$. Assume both $h_0$ and $v_0$ are positive constants. The height of the ball at time $t$ is given by the formula below:

$$h(t) = h_0 + v_0 t - (1/2)gt^2$$

(a) When does the ball reach its highest point?

(b) How high is it at that point ?

(c) What is the instantaneous velocity of the ball at its highest point ?

(d) What is the average velocity of the ball for the time interval $0 \leq t \leq 1$ assuming that it is in the air during this whole time interval ?

**QUESTION 37:**

Find the dimensions of the cylinder of maximal volume that will just exactly fit inside a sphere of radius $R$.