Math 528: Algebraic Topology Class Notes

Lectures by Denis Sjerve, notes by Benjamin Young

Term 2, Spring 2005

Contents

Chapter 1

January 4

1.1 A Rough Definition of Algebraic Topology

Algebraic topology is a formal procedure for encompassing all functorial relationships between the worlds of topology and algebra:

$$\boxed{\text{world of topological problems}} \xrightarrow{F} \boxed{\text{world of algebraic problems}}$$

Examples:

1. The retraction problem: Suppose X is a topological space and $A \subseteq X$ is a subspace. Does there exist a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$? r is called a *retraction* and A is called a *retract* of X. If a retraction $\exists$ then we have a factorization of the identity map on $A : A \xrightarrow{i} X \xrightarrow{r} A,$ where $r \circ i = id_A$.

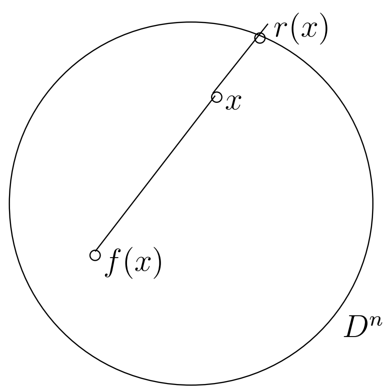
   Functoriality of F means that the composite $F(A) \xrightarrow{F(i)} F(X) \xrightarrow{F(r)} F(A)$ (respectively $F(A) \xrightarrow{F(r)} F(X) \xrightarrow{F(i)} F(A)$) is the identity on $F(A)$ if F is a covariant (respectively contravariant) functor. As an example consider the retraction problem for X the n-disk and A its boundary, $n > 1$: $S^{n-1} = \partial(D^n) \xrightarrow{i} D^n \xrightarrow{r} \partial(D^n) = S^{n-1}$.

Suppose that the functor F is the nth homology group:

$$\begin{array}{ccccc} H_{n-1}(\partial(D^n)) & \xrightarrow{i_*} & H_{n-1}(D^n) & \xrightarrow{r_*} & H_{n-1}(\partial D^n) \\ || & & || & & || \\ \mathbb{Z} & \xrightarrow{i_*} & 0 & \xrightarrow{r_*} & \mathbb{Z} \end{array}$$

Such a factorization is clearly not possible, so ∂D^n is not a retract of D^n

2. When does a self map $f : X \to X$ have a fixed point? That is, when does $\exists\ x \in X$ such that $f(x) = x$? For example suppose $f : X \to X$, where $X = D^n$. Assume that $f(x) \neq x$ for all $x \in D^n$. Then we can project $f(x)$ through x onto a point $r(x) \in \partial D^n$, as follows:



Then $r : D^n \to \partial D^n$ is continuous and $r(x) = x$ if $x \in \partial D^n$. Thus r is a retraction of D^n onto its boundary, a contradiction. Thus f must have a fixed point.

3. What finite groups G admit fixed point free actions on some sphere S^n? That is, when does $\exists$ a map $G \times S^n \to S^n$, $(g, x) \mapsto g \cdot x$, such that $h \cdot (g \cdot x) = (hg) \cdot x$, $id \cdot x = x$, and for any $g \neq id$, $g \cdot x \neq x$ for all $x \in S^n$.

This is "still" unsolved (although some of the ideas involved in the supposed proof of the Poincaré conjecture would do it for dimension 3). However, lots is known about this problem.

For example, any cyclic group $G = Z_n$ admits a fixed-point free action on any odd-dimensional sphere:

$$S^{2k-1} = \{(z_1, \ldots, z_k) \subseteq \mathbb{C}^k | \sum z_i \bar{z}_i = 1\}.$$

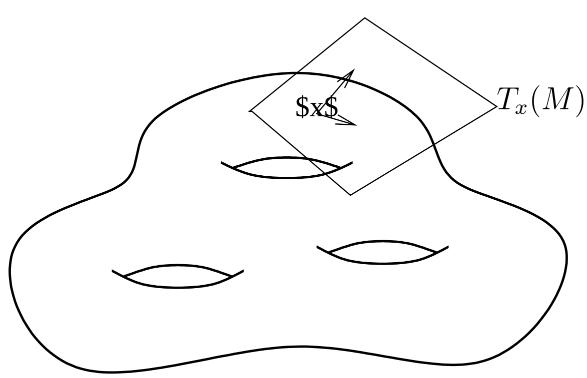
A generator for G is $T : S^1 \to S^1$, $T(x) = \xi x$, where $\xi = e^{2\pi i/n}$. Then a fixed point free action of G on S^{2k-1} is given by

$$T(z_1, \ldots, z_k) = (\xi z_1, \ldots, \xi z_k).$$

There are other actions as well.

Exercise: Construct some other fixed point free actions of G on S^{2k-1}.

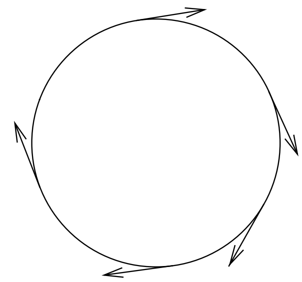
4. Suppose M^n is a smooth manifold of dimension n. What is the span of M, that is what is the largest integer k such that there exists a k-plane varing continuously with respect to x? This means that at each point $x \in M$ we have k linearly independent tangent vectors $v_1(x), \ldots, v_k(x)$ in T_xM, varying continuously with respect to x.



Definition: if $k = n$ then we say that M is *parallelizable.*

In all cases $k \leq n$.

In the case of the 2-sphere we can't find a non-zero tangent vector which varies continuously over the sphere, so $k = 0$. This is the famous "fuzzy ball" theorem. On the other hand S^1 is parallelizable.



S^3 is also parallizable. To see this consider $\mathbb{R}^4$ with basis the unit quaternions $1, i, j, k$. Thus a typical quaternion is $q = q_0 + q_1 i + q_2 j + q_3 k$, where the q_i are real. R^4 becomes a division algebra, where we multiply quaternions using the rules

$$i^2 = j^2 = k^2 = -1, ij = k, ji = -k, jk = i, kj = -i, ki = j, ik = -j$$

and the distributive law. That is

$$qq' = (q_0 + q_1 i + q_2 j + q_3 k)(q'_0 + q'_1 i + q'_2 j + q'_3 k) = r_0 + r_1 i + r_2 j + r_3 k$$

where

$$\begin{aligned} r_0 &= q_0 q'_0 - q_1 q'_1 - q_2 q'_2 - q_3 q'_3 \\ r_1 &= q_0 q'_1 + q_1 q'_0 + q_2 q'_3 - q_3 q'_2 \\ r_2 &= q_0 q'_2 - q_1 q'_3 + q_2 q'_0 + q_3 q'_1 \\ r_3 &= q_0 q'_3 + q_1 q'_2 - q_2 q'_1 + q_3 q'_0 \end{aligned}$$

The conjugate of a quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ is defined by $\bar{q} = q_0 - q_1 i - q_2 j - q_3$. It is routine to show that $q\bar{q} = \sum_n q_n^2$. We define the norm of a quaternion by $|q| = \sqrt{q\bar{q}}$. Then $|qq'| = |q||q'|$

The space of unit quaternions

$$\{q_0 + q_1 i + q_2 j + q_3 k | \sum_n q_n^2 = 1\}$$

is just the 3-sphere, and it is a group. Pick three linearly independent vectors at some fixed point in S^3. Then use the group structure to translate this frame to all of S^3.

5. The homeomorphism problem. When is X homeomorphic to Y?

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "F"'] & Y \arrow[d, "F"'] \\ F(x) \arrow[r, "F(f)"] & F(Y) \end{tikzcd}$$

6. The homotopy equivalence problem. When is X homotopically equivalent to Y?

7. The lifting problem. Given $X \xrightarrow{f} B$ and $E \xrightarrow{p} b$, can we find a map $\tilde{f} : X \to E$ such that $p\tilde{f} \simeq f$?

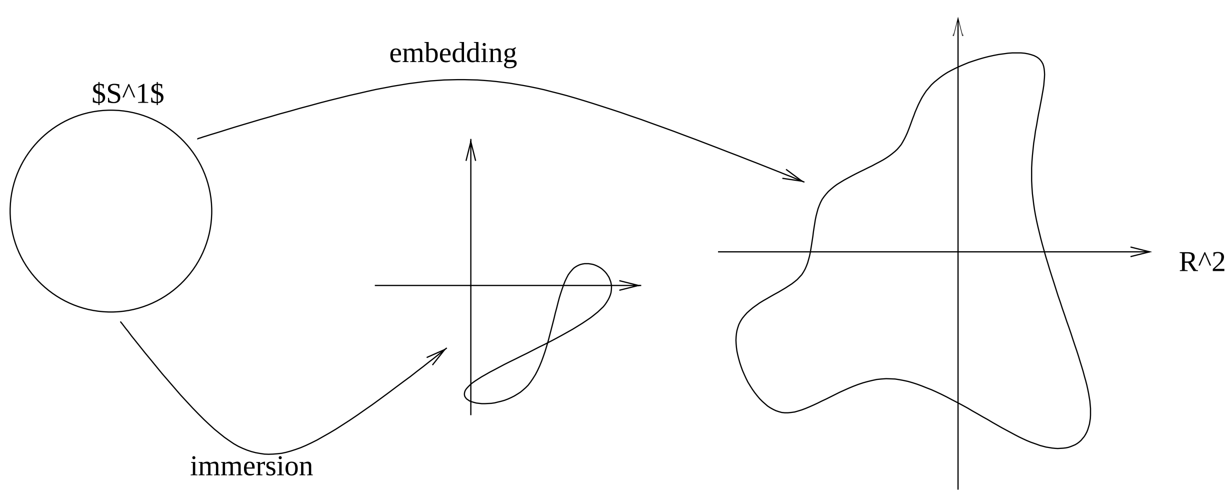
8. The embedding problem for manifolds. What is the smallest k such that the n-dimensional manifold M can be embedded into $\mathbb{R}^{n+k}$?

   Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and $\mathbb{R}P^n$ = real projective space of dimension n:
   $$\mathbb{R}P^n \overset{\text{def}}{=} S^n/x \sim -x.$$

   Alternatively, $\mathbb{R}P^n$ is the space of lines through the origin in $\mathbb{R}^{n+1}$.

   Unsolved problem: what is the smallest k such that $\mathbb{R}P^n \subseteq \mathbb{R}^{n+k}$?

9. Immersion problem: What is the least k such that $\mathbb{R}P^n$ immerses into $\mathbb{R}^{n+k}$?



10. The computation of homotopy groups of spheres.

$$\pi_k(X) \stackrel{\text{def}}{=} \text{ the set of homotopy classes of maps } f : S^k \to X.$$

It is known that $\pi_k(X)$ is a group $\forall\ k \geq 1$ and that $\pi_k(X)$ is abelian $\forall k \geq 2$. What is $\pi_k(S^n)$? The Freudenthal suspension theorem states that $\pi_k(S^n) \approx \pi_{k+1}(S^{n+1})$ if $k < 2n - 1$. For example,

$$\pi_4(S^3) \approx \pi_5(S^4) \approx \pi_6(S^5) \approx \cdots .$$

We know that these groups are all $\approx \mathbb{Z}_2$ and $\pi_3(S^2) = \mathbb{Z}$.

Chapter 2

January 6

2.1 The Mayer-Vietoris Sequence in Homology

Recall the van Kampen Theorem: Suppose X is a space with a base point x_0, and X_1 and X_2 are path connected subspaces such that $x_0 \in X_1 \cap X_2$, $X = X_1 \cup X_2$ and $X_1 \cap X_2$ is path connected. Consider the diagram

$$\begin{array}{ccc} X_1 \cap X_2 & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow{j_2} & X \end{array}$$

Apply the fundamental group 'functor' π_1 to this diagram:

$$
\begin{array}{ccc}
\pi_1(X_1 \cap X_2) & \xrightarrow{i_{1\#}} & \pi_1(X_1) \\
{\scriptstyle i_{2\#}}\downarrow & & \downarrow{\scriptstyle j_{1\#}} \\
\pi_1(X_2) & \xrightarrow{j_{2\#}} & \pi_1(X)
\end{array}
$$

Question: How do we compute $\pi_1(X)$ from this data?

There exists a group homomorphism from the free product $\pi_1(X_1) * \pi_1(X_2)$ into $\pi_1(X)$, given by $c_1 \cdot c_2 \mapsto j_{1\#}(c_1) \cdot j_{2\#}(c_2)$.

Fact: This map is onto $\pi_1(X)$. However, there exists a kernel coming from $\pi_1(X_1 \cap X_2)$. In fact, $i_{1\#}(\alpha) \cdot i_{2\#}(\alpha^{-1})$, for every $\alpha \in \pi_1(X_1 \cap X_2)$, is in the kernel because $j_{1\#}i_{1\#} = j_{2\#}i_{2\#}$.

Theorem: (van Kampen): Suppose all the spaces $X_1, X_2, X_1 \cap X_2$ contain the base point $x_0 \in X = X_1 \cup X_2$, and every space is path connected. Then $\pi_1(X) \approx \pi_1(X_1) * \pi_2(X_2)/K$ where K is the normal subgroup generated by all elements of the form $i_{1\#}(\alpha) \cdot i_{2\#}(\alpha^{-1})$, where $\alpha \in \Pi_2(X_1 \cap X_2)$.

Definition: Let X be a space with a base point $x_0 \in X$. The *nth homotopoy group* is the set of all homotopy classes of maps $f : (I^n, \partial I^n) \to (X, x_0)$. Here,

$$
\begin{aligned}
I^n &= \{(t_1, \ldots, t_n) | 0 \leq t_i \leq 1\}; \\
\partial I^n &= \text{the boundary of } I^n \\
&= \{(t_1, \ldots, t_n) \mid 0 \leq t_i \leq 1, \text{ some } t_i = 0 \text{ or } 1\}
\end{aligned}
$$

Notation: $\pi_n(X, x_0) = \pi_n(X) =$ the nth homotopy group.

Fact: $I^n/\partial I^n \approx S^n$. Therefore, $\Pi_n(X)$ consists of the homotopy classes of maps $f : (S^n, *) \to (X, x_0)$.

Question: Is there a van Kampen theorem for Π_n?

Answer: **NO**.

But there is an analogue of the van Kampen Theorem in *Homology*: it is the

Meyer–Vietoris sequence. Here is the setup:

$$\begin{array}{ccc} X_1 \cap X_2 & \xrightarrow{i_1} & X_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ X_2 & \xrightarrow{j_2} & X = X_1 \cup X_2 \end{array}$$

Question: What is the relationship amongst $H_*(X_1 \cap X_2)$, $H_*(X_1)$, $H_*(X_2)$ and $H_*(X)$?

Theorem: (Mayer-Vietoris) Assuming some mild hypotheses on X_1, X_2, X there exists a long exact sequence:

$$\begin{array}{rcl} \cdots \to H_n(X_1 \cap X_2) & \overset{\alpha_*}{\to} & H_n(X_1) \oplus H_n(X_2) \overset{\beta_*}{\to} H_n(X) \overset{\partial}{\to} \\ H_{n-1}(X_1 \cap X_2) & \overset{\alpha_*}{\to} & \cdots \to H_0(X) \to 0. \end{array}$$

The maps α_* and β_* are defined by

$$\beta_* : H_n(X_1) \oplus H_n(X_2) \to H_n(X),\ \beta_*(c_1, c_2) \to j_{1*}(c_1) + j_{2*}(c_2)$$
$$\alpha_* : H_n(X_1 \cap X_2) \to H_n(X_1) \oplus H_n(X_2),\ c \to (i_{1\#}(c), -i_{2\#}(c))$$

The minus sign gets included in α_* for the purpose of making things exact (so that $\beta_*\alpha_* = 0$). One could have included it in the definition of β_* instead and still be correct.

Proof: There exists a short exact sequence of chain complexes

$$0 \to C_*(X_1 \cap X_0) \overset{\alpha}{\to} C_*(X_1) \oplus C_*(X_2) \overset{\beta}{\to} C_*(X_1 + X_2) \to 0$$

where $C_n(X_1 + X_2)$ is the group of chains of the form $c_1 + c_2$, where c_1 comes from X_1 and c_2 comes from X_2. The 'mild hypotheses' imply that the inclusion $C_*(X_1 + X_2) \subseteq C_*(X)$ is a chain equivalence.

Lemma: If

$$0 \to C'' \overset{\alpha}{\to} C' \overset{\beta}{\to} C \to 0$$

is an exact sequence of chain complexes, then there exists a long exact sequence

$$\cdots \to H_n(C'') \xrightarrow{\alpha_*} H_n(C') \xrightarrow{\beta_*} H_n(C) \xrightarrow{\partial} H_{n-1}(C'') \xrightarrow{\alpha_*} \cdots$$

To prove this, one uses the "snake lemma" which may be found in Hatcher, or probably in most homological algebra references.

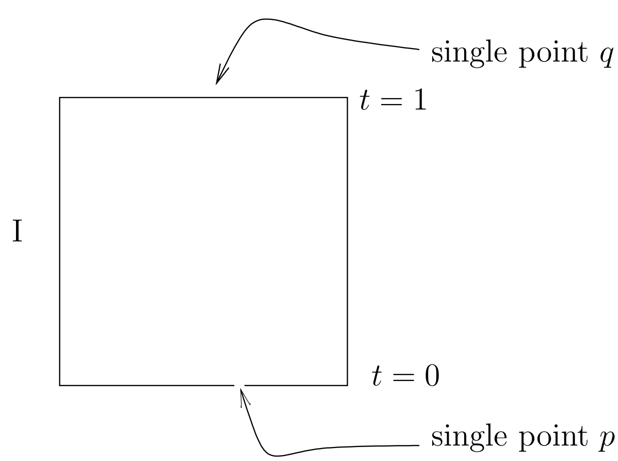
Remarks: There exists a Mayer-Vietoris sequence for reduced homology, as well:

$$\cdots \to \tilde{H}_n(X_1 \cap X_2) \xrightarrow{\alpha_*} \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \xrightarrow{\beta_*} \tilde{H}_n(x) \xrightarrow{\partial} \tilde{H}_{n-1}(X_1 \cap X_2) \xrightarrow{\alpha_*} \cdots$$

The reduced homology groups are defined by $\tilde{H}_n \overset{def}{=} H_n(X, x_0)$. Therefore $H_n(X) \approx \tilde{H}_n(X)$ for $n \neq 0$ and $H_0(X) \approx \tilde{H}_0(X) \oplus \mathbb{Z}$.

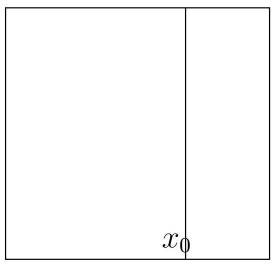
Examples. The unreduced suspension of a space X is

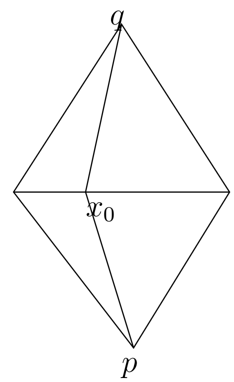
$$SX := X \times [0,1]/(x \times 0 = p, x \times 1 = q,\ \forall x \in X)$$



We also have the reduced suspension for a space X with a base point x_0:

$$\Sigma X = SX/(x_0 \times [0,1])$$

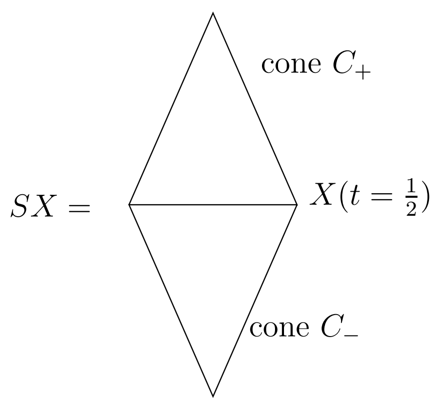




Fact: Suppose $A \subseteq W$ is a contractible subspace. Then, assuming certain mild hypotheses, $W \to W/A$ is a homotopy equivalence.

Corollary: $\tilde{H}_n(SX) \xrightarrow{\approx} \tilde{H}_{n-1}(X)$.

Proof:



Consider the Mayer-Vietoris sequence for the pair (C_+, C_-):

$$\cdots \to \tilde{H}_n(X) \xrightarrow{\alpha_*} \underbrace{\tilde{H}_n(C_+)}_{=0} \oplus \underbrace{\tilde{H}_n(C_-)}_{=0} \xrightarrow{\beta_*} \tilde{H}_n(SX) \xrightarrow{\delta} \tilde{H}_{n-1}(X) \xrightarrow{\alpha_*} \cdots$$

$\square$

Corollary $\tilde{H}_n(S^k) \sim \tilde{H}_{n-1}(S^{k-1})$.

Pf. $S(S^{k-1}) = S^k$. $\square$

2.2 Example: Two Spaces with Identical Homology

Recall that real projective n-space is $\mathbb{R}P^n = S^n/(x \sim -x)$, the n-sphere with antipodal points identified. Let us write $S^2(\mathbb{R}P^2)$ for $S(S(\mathbb{R}P^2))$. Define

$$\begin{aligned} X &\stackrel{def}{=} \mathbb{R}P^2 \vee S^2(\mathbb{R}P^2) \\ Y &\stackrel{def}{=} \mathbb{R}P^4, \end{aligned}$$

where $A \vee B$ is the one point union of A, B. Now,

$$H_i(Y) = H_i(\mathbb{R}P^4) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}_2 & \text{if } i = 1, 3 \\ 0 & \text{otherwise.} \end{cases}$$

Exercise Show that $\tilde{H}_i(A \vee B) \approx \tilde{H}_i(A) \oplus \tilde{H}_i(B)$ using an appropriate Mayer-Vietoris sequence.

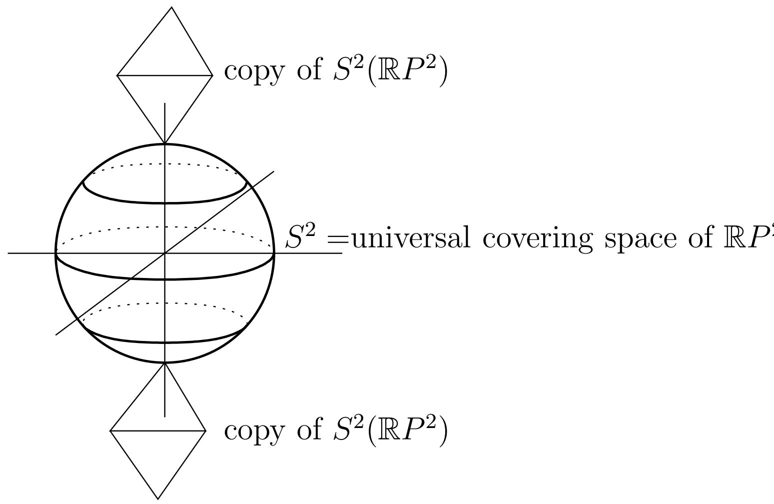
So we can compute that

$$H_i(X) = H_i(\mathbb{R}P^2 \vee S^2(\mathbb{R}P^2)) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}_2 & \text{if } i = 1, 3 \\ 0 & \text{otherwise,} \end{cases}$$

which is the same homology as Y.

Is it the case that X and Y are "the same" in some sense? Perhaps "same" means "homeomorphic"? But $Y = \mathbb{R}P^4$ is a 4–dimensional manifold, whereas $X = \mathbb{R}P^2 \vee S^2(\mathbb{R}P^2)$ is not a manifold. So X is not homeomorphic to Y.

Can "same" mean "homotopy equivalent?" Still no. The universal covering space of Y is S^4, whereas the universal covering space of X is:



If X and Y were homotopically equivalent then their universal covering spaces would also be homotopically equivalent. Let $\tilde{X}$ and $\tilde{Y}$ be the universal covering space. Then $H_2(\tilde{X}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, but $H_2(\tilde{Y}) = 0$.

Question: Does there exist a map $f : X \to Y$ (or $g : Y \to X$) such that f_* (resp, g_*) is an isomorphism in homology?

Again, the answer is no, and we shall see why next week.

Chapter 3

January 11

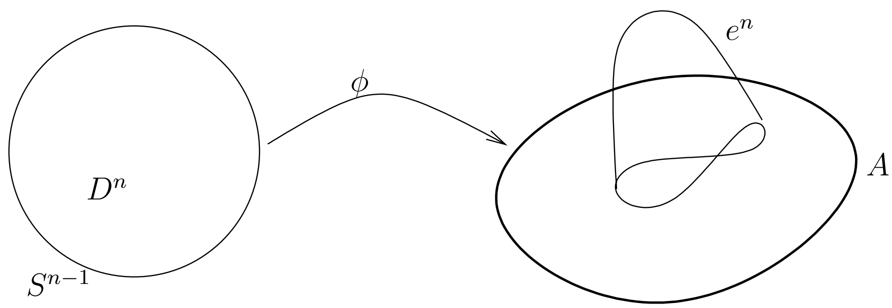
3.1 Hatcher's Web Page

Hatcher's web page is: `http://www.math.cornell.edu/~hatcher`. There, you can find an electronic copy of the text.

3.2 CW Complexes

The fundamental construction is attaching an n-cell e^n to a space A. Suppose we have a map $\phi : S^{n-1} \to A$. In general we can't extend this to a map $D^n \xrightarrow{\Phi} A$, but we can extend it if we enlarge the space A to X, where

$$X = A \sqcup D^n/(\phi(x) \sim x \ \forall x \in \partial D^n = S^{n-1}.$$

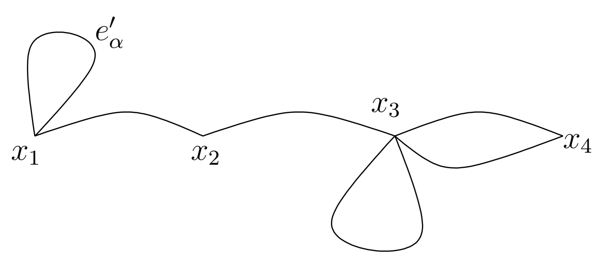


We say that X is obtained from A by attaching an n-cell e^n. The given map $\phi : S^{n-1} \to A$ is the sattaching map and its extension $\Phi : D^n \to X$ is called the characteristic map.

Definition: of a CW complex: $X = X^0 \cup X^1 \cup \cdots$

Start with a discrete set of points $X^0 = \{x_1, x_2, \ldots\}$. Now attach 1-cells via maps $\phi_\alpha : S^0 \to X^0$, where $\alpha \in A =$ some index set.

X^1 is the result of attaching 1-cells.



Suppose we have constructed X^{n-1}, the $(n-1)$-skeleton. Then X^n is the result of attaching n-cells to X^{n-1} by maps $\phi_\beta : S^{n-1} \to X^{n-1}$, $\beta \in B$:

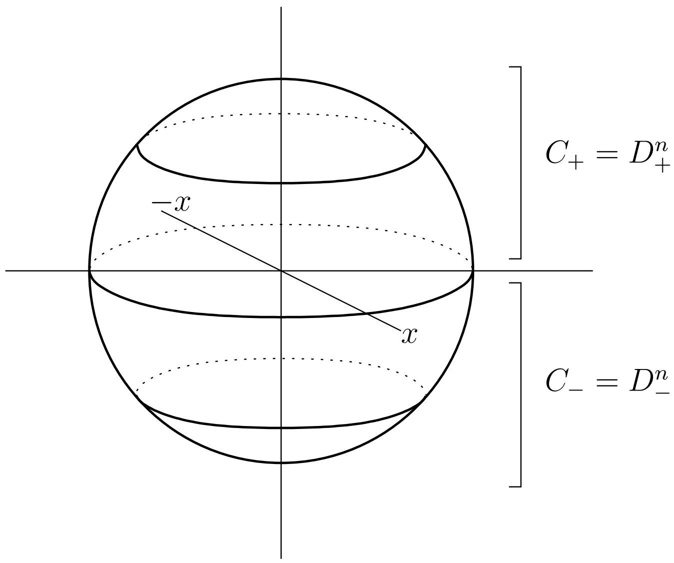
$$X^n := X^{n-1} \sqcup_{\beta \in B} D^n_\beta \Big/ (x \sim \phi_\beta(x),\ \forall x \in \partial D^n_\beta \text{ and } \ \beta \in B)$$

Examples.

1. $X = S^n = \mathrm{pt} \cup e^n = e^0 \cup e^n$. $A = \mathrm{pt} = e^0$. $\phi : S^{n-1} \to A$ is the constant map. Thus, $X = A \sqcup D^n \big/ x \in \partial D^n, x \sim e^0$. Thus X is the n-sphere S^n.

2. $S^n = \partial(\Delta^{n+1})$, where Δ^{n+1} is the standard $n+1$-simplex. For example, the surface of the tetrahedron is $\partial(\Delta^3) = S^2$. To describe S^2 this way (as a simplicial complex) we need: 4 vertices, 6 edges and 4 faces. This is far less efficient than the CW-complex description above.

3. $\mathbb{R}P^n :=$ the space of lines through the origin in $\mathbb{R}^{n+1} = S^n \big/ (x \sim -x)$. We can think of a point in $\mathbb{R}P^n$ as a pair of points $\{x, -x\}, x \in S^n$.



There is a double covering $\phi : S^n \to \mathbb{R}P^{n-1}$, where $\phi(x) = (x, -x)$, and $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup_\phi e^n$. Therefore $\mathbb{R}P^n$ has a cell decomposition of the form $\mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n$.

4. $\mathbb{C}P^n$, complex projective n-space, is the space of one-dimensional compex subvector spaces of $\mathbb{C}^{n+1}$. It is homeomorphic to the quotient space

$$S^{2n+1} \big/ (x \sim \zeta x,\ \forall\ x \in S^{2n+1} \text{ and all unit complex numbers } \zeta).$$

Here is another way to describe $\mathbb{C}P^n$: there exists an action of S^1 on S^{2n+1}. Let us think of S^{2n+1} in the following way:

$$S^{2n+1} = \{(z_1, \ldots, z_{n+1}) \mid z_1\bar{z}_1 + \cdots + z_{n+1}\bar{z}_{n+1} = 1\}.$$

Then S^1 acts on S^{2n+1} by $\zeta \cdot (z_1, \ldots, z_{n+1}) = (\zeta z_1, \ldots, \zeta z_{n+1})$.

Exercie: Let $\phi : S^{2n+1} \to S^{2n+1}/S^1 = \mathbb{C}P^n$ be the quotient map. Verify that $\mathbb{C}P^{n+1} = \mathbb{C}P^n \cup_\phi e^{2n+2}$. Thus $\mathbb{C}P^n$ has a cell decomposition of the form $\mathbb{C}P^n = e^0 \cup e^2 \cup e^4 \cup \cdots \cup e^{2n}$.

Remark: Suppose $X = A \cup_\phi e^n$, for some $\phi : S^{n-1} \to A$. Then there is an inclusion map $i : A \subseteq X$ and $X/A = D^n/\partial D^n = S^n$. More generally, let X be a CW complex with n cells $e^n_\alpha, \alpha \in A$. Then

$$X^n/X^{n-1} = \bigvee_{\alpha \in A} S^n_\alpha.$$

3.3 Cellular Homology

Let X be a CW complex. We have a commutative diagram:

$$\begin{tikzcd}
H_{n+1}(X^{n+1}, X^n) \arrow[rr, "d_{n+1}"] \arrow[dr, "\delta"'] & & H_n(X^n, X^{n-1}) \arrow[rr, "d_n"] \arrow[dr, "\delta"] & & H_n - 1(X^{n-1}, X^{n-2}) \arrow[r] & \cdots \\
& H_n(X^n) \arrow[ur, "i_*"] \arrow[rr, "0"] & & H_{n-1}(X^{n-1}) \arrow[ur, "i_*"] & &
\end{tikzcd}$$

This diagram serves to define the maps d_n. Observe that $d_n d_{n+1} = 0$, so we have a chain complex , the so-called cellular chain complex of X:

$$C_*(X) = \{C_n(X) = H_n(X^n, X^{n-1}), d_n\}$$

.

The homology of this chain complex is called cellular homology.

Definition: $H_*(C_*(X)) = H_*^{CW}(X)$.

Theorem: $H_*^{CW}(X)$ is naturally isomorphic to $H_*(X)$.

Examples.

1. $S^n = e^0 \cup e^n$ $X^0 = pt$, $X^n = S^n$ and $X^1, X^2, \cdots, X^{n-1}$ are empty. Thus

$$C_i(X) = \begin{cases} Z & i = 0, n \\ 0 & \text{otherwise} \end{cases}$$

   If $n > 1$, then all $d_k = 0$ and

$$H_i^{CW}(S^n) = \begin{cases} Z & i = 0, n \\ 0 & \text{otherwise} \end{cases}$$

2. $\mathbb{C}P^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$. Therefore, all d_k are 0 and so

$$H_i^{CW}(\mathbb{C}P^n) = \begin{cases} Z & i = 0, 2, 4, \cdots, 2n \\ 0 & \text{otherwise} \end{cases}$$

3. Let $X = \mathbb{C}P^n$ and $Y = S^2 \vee S^4 \vee \cdots S^{2n}$, where all attaching maps for Y are trivial. (i.e. constant). It is easy to check that

$$H_i(Y) = \begin{cases} Z & i = 0, 2, 4, \cdots, 2n \\ 0 & \text{otherwise} \end{cases}$$

   That is, $H_*(X) \approx H_*(Y)$. But X is not homeomorphic to Y if $n > 1$. X is a compact $2n$-dimensional manifold with no boundary; Y is not even a manifold if $n > 1$.

Exercie: Show that $\mathbb{C}P^1 \approx S^2$.

Question: if $n > 1$ is there a map $f : X \to Y$ inducing an isomorphism on H_*?

Answer: No, but one needs cohomology in order to see it.

3.4 A Preview of the Cohomology Ring

The cohomology groups of a space X (whatever they are) actually form a graded ring. The cohomology ring structure of $X = \mathbb{C}P^n$ is:

$$H^*(X) \approx \mathbb{Z}[x]/(x^{n+1}), \text{ where } x \in H^2(\mathbb{C}P^n) \approx \mathbb{Z} \text{ is a generator.}$$

Here, we have $1 \in H^0(X) \approx \mathbb{Z}$, $x \in H^2(X) \approx \mathbb{Z}$, $x^2 \in H^4(X) \approx \mathbb{Z}$, $\ldots$, $x^n \in H^{2n}(X) \approx \mathbb{Z}$. Moreover, $H^i(X) = 0$ if i is odd.

The answer for Y is:

$$H^i(Y) = \begin{cases} \mathbb{Z} & i = 0, 2, \ldots, 2n \\ 0 & \text{otherwise} \end{cases}.$$

This is the same information as far as homology is concerned. However, all "products" turn out to be zero in $H^*(Y)$. So the cohomology rings of these two spaces are not isomorphic, and therefore there is no map $f : X \to Y$ inducing an isomorphism on H_*.

3.5 Boundary Operators in Cellular Homology

Suppose X a CW-complex and $C_*(X)$ is its cellular chain complex. Then

$$C_n(X) = H_n(X^n, X^{n-1}) \approx H_n(X^n/X^{n-1}, *) \approx \tilde{H}_n\left(\bigvee_{\alpha \in A} S^n_\alpha\right).$$

Thus $C_n(X)$ is a free abelian group with one generator for each n-cell e^n_α. Therefore, we can identify the generators with the n-cells e^n_α.

The boundary operator $d_n : H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ will be given by an equation of the form

$$d_n(e^n_\alpha) = \sum_{\beta \in B} d_{\alpha\beta} e^{n-1}_\beta, \text{ where the } n-1\text{-cells are } e^{n-1}_\beta \text{ for } \beta \in B.$$

Let $\phi_\alpha : S^{n-1} \to X^{n-1}$ be an attaching map for e^n_α. Consider the composite

$$f_{\alpha\beta} : S^{n-1} \xrightarrow{\phi_\alpha} X^{n-1} \xrightarrow{c} X^{n-1}/X^{n-2} \approx \bigvee_{\beta \in B} S^{n-1}_\beta \xrightarrow{c} S^{n-1}_\beta = S^{n-1},$$

where the maps labelled c are collapsing maps.

Definition: If $f : S^m \to S^m$ then $f_* : H_m(S^m) \approx \mathbb{Z} \to H_m(S^m) \approx \mathbb{Z}$ is multiplication by some integer k. The $\deg f$ is defined to be k.

Theorem: The degree of $f_{\alpha\beta}$ is $d_{\alpha\beta}$.

Chapter 4

January 13

4.1 Homology of $\mathbb{R}P^n$

Today's goal will be to compute $H_*(\mathbb{R}P^n)$ using cellular homology. Recall that $\mathbb{R}P^n = \underbrace{\underbrace{e^0 \cup e^1}_{S^1} \cup e^2}_{\mathbb{R}P^2} \cup \cdots \cup e^n$.

We define $\mathbb{R}P^\infty = \lim_{n\to\infty} \mathbb{R}P^n = S^\infty / x \sim -x$.

The cellular chain groups of a CW complex X are $C_n(X) = H_n(X^n, X^{n-1})$. Thus $C_n(X)$ is a free abelian group with generators in 1-1 correspondence with the n-cells in X.

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_0 \to 0$$

In the case of $\mathbb{R}P^n$, it turns out that $C_k \approx \mathbb{Z}\ \forall k$ and the boundary operators are alternatively multiplication by 0 and multiplication by 2 :

$$\cdots \to 0 \to C_n \to \cdots \xrightarrow{\times 2} C_5 \xrightarrow{0} C_4 \xrightarrow{\times 2} C_3 \xrightarrow{0} C_2 \xrightarrow{\times 2} C_1 \xrightarrow{0} C_0 \to 0$$

The boundary operator $C_n \xrightarrow{d_n} C_{n-1}$ will be 0 if n is odd and multiplication by 2 otherwise.

Corollary:

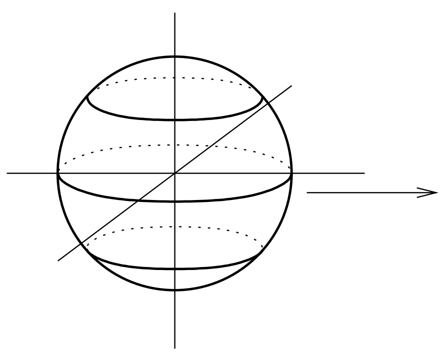
$$H_i(\mathbb{R}P^n) = \begin{cases} Z & i = 0 \\ Z_2 & i = 1, 3, 5, \cdots, 2k-1 < n \\ 0 & i = 2, 4, \cdots, 2k < n \end{cases}$$

and

$$H_n(\mathbb{R}P^n) = \begin{cases} 0 & n \text{ even} \\ \mathbb{Z} & n \text{ odd} \end{cases}$$

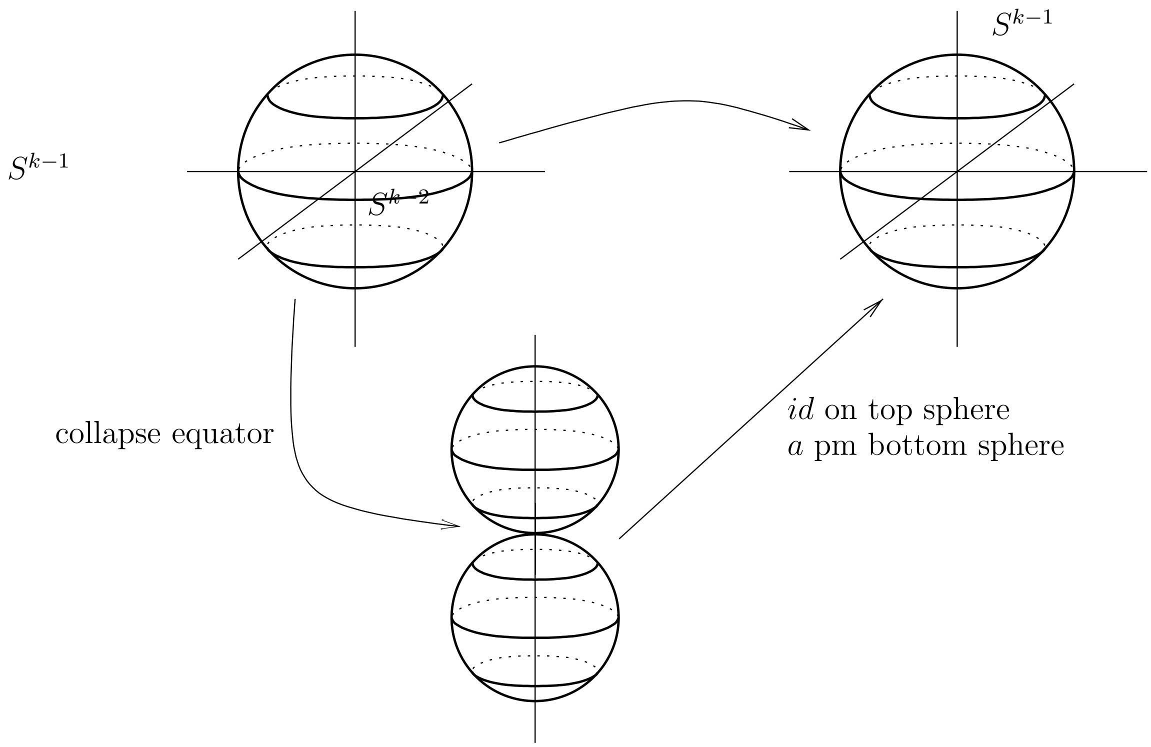
Description of $d_k : C_k \to C_{k-1}$

S^{k-1}



$\mathbb{R}P^{k-1} \longrightarrow \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} = S^{k-1}$

Note that $\mathbb{R}P^{k-2}$ comes from the equator of the sphere, and is collapsed to a point:



where $a : S^{k-1} \to S^{k-1}$ is the antipodal map. Therefore, $d_k : C_k \to C_{k-1}$ is $Z \xrightarrow{\times j} \mathbb{Z}$, where $j = deg(id + a)$. But

$$\deg(id + a) = 1 + (-1)^k = \begin{cases} 2 & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

4.2 A Pair of Adjoint Functors

Y^I is the space of mappings $I = [0, 1] \to Y$, with the compact open topology. If Y has a base point y_0 then we can define the *loop space* by

$$\Omega Y \overset{def}{=} \text{ the space of maps } I \xrightarrow{\omega} Y \text{ such that } \omega(0) = \omega(1) = y_0.$$

We say that the suspension functor Σ and the loop space functor Ω are adjoint because there is a natural isomorphism of sets:

$$\text{Maps}(\Sigma X, Y) \approx \text{Maps}(X, \Omega Y),\ \ f(x,t) = f'(x)(t)$$

where $f : \Sigma X \to Y$ and $f' : X \to \Omega Y$. This correspondence induces a natural equivalence on homotopy: $[\Sigma X, Y] \approx [X, \Omega Y]$.

ΩY has extra structure: it is an H–space. H probably stands for either "homotopy" or "Heinz Hopf", depending on whether you are Heinz Hopf or not.

ΩY is a group up to homotopy. The group multiplication $\mu : \Omega Y \times \Omega Y \to \Omega Y$ is concatenation of paths:

$$\mu(\omega, \eta) = \begin{cases} \omega(2t), 0 \le t \le \frac{1}{2} \\ \eta(2t-1), \frac{1}{2} \le t \le 1 \end{cases}$$

We write $\mu(\omega, \eta) = \omega * \eta$.

Exercise: Let ϵ be the constant map at y_0. Then show that $\epsilon * \omega \simeq \omega * \epsilon \simeq \omega$. The inverse of ω is the path $\omega^{-1}(t) = \omega(1-t)$. That is, $\omega * \omega^{-1} \simeq \epsilon \simeq \omega^{-1} * \omega$. The operation is associative.

Therefore, $[X, \Omega Y]$ is an actual group, so $[\Sigma X, Y]$ is also a group. Here, the group operation is defined as follows: if $f : \Sigma X \to Y$, $g : \Sigma X \to Y$ then

$$f \cdot g(x,t) = \begin{cases} f(x, 2t) & 0 \le t \le \frac{1}{2} \\ g(x, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Examples:

1. Let $f : S^1 \to S^1$ be $f(z) = z^k$. Then $\deg f = k$.

2. We wish to define a map $g : S^n \to S^n$ with $\deg g = k$.

   Definition: Suppose $f : X \to Y$ is base point preserving. Then we can define $\Sigma f : \Sigma X \to \Sigma Y$ by $\Sigma f(x,t) = (f(x), t)$.

   Now, suspending $f : z \mapsto z^k$ $n-1$ times gives a map $\Sigma^{n-1} f : S^n \mapsto S^n$ of degree k.

Chapter 5

January 18

5.1 Assignment 1

p.155 #1,3,5,7,9(ab); p156#12, p.257 #19

Here are some comments:

(1) paraphrased is: show that $f : D^n \to D^n$ has a fixed point. This is the Brouwer Fixed point theorem and we have proved it already. The question asks us to prove it in a special way: apply degree theory to the map that sends both northern and southern hemispheres to the southern hemisphere by f.

(3) No comment.

(5) Let r_1, r_2 of S^n be reflections through hyperplanes. Show that $r_1 \simeq r_2$ through a homotopy consisting of reflections through hyperplanes.

(7) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear transformation.Then we have

a commutative diagram

$$\begin{array}{ccc} \tilde{H}_n(\mathbb{R}^n, R^n - \{0\}) & \xrightarrow{f_*} & \tilde{H}_n(\mathbb{R}^n, R^n - \{0\}) \\ \approx\big\downarrow & & \approx\big\downarrow \\ \mathbb{Z} \approx \tilde{H}_n(R^n - \{0\}) & \xrightarrow{f_*} & \tilde{H}_n(R^n - \{0\}) \approx \mathbb{Z} \end{array}$$

Show that $f_* = id$ if and only if $det f > 0$. Use linear algebra, Gaussian reduction, etc.

(9ab) Compute the homology groups of:

(a) $S^2 \big/$(north pole = south pole.)

(b) $S^1 \times (S^1 \vee S^2)$. Cellular homology is probably the best approach.

(12) Show that the map $S^1 \times S^1 \to \left(S^1 \times S^1 \big/ S^1 \vee S^1\right) \approx S^2$ is not homotopic to a point.

(19) Compute the homology of thetruncated projective space $\mathbb{R}P^n \big/ \mathbb{R}P^m$, $m < n$.

5.2 Homology with Coefficients

Example: $f : X \to \mathbb{R}P^\infty = S^\infty / x \sim -x$. For a model for S^∞ we use the space of sequences

$$(x_1, x_2, x_3, \ldots)$$

with $x_i \in \mathbb{R}$ such that $\sum_{i=1}^\infty x_i^2 = 1$ and $x_k = 0$ for $k \gg 0$ (means: k sufficiently large).

We will see that $\exists$ natural equivalences

$$[X, \mathbb{R}P^\infty] \approx H^1(X; Z_2) \approx \text{Hom } (H_1(X; Z), Z_2).$$

Thus $\exists$ a unique element in Hom $(H_1(X; Z), Z_2)$ corresponding to the homotopy class of f. Which leads us to the question: how does one put coefficients into a homology theory? There is a natural way to do this.

Let X be some space and $C_*(X)$ one of the chain complexes associated to X. A typical n-chain is

$$\sum_i n_i \sigma_i, \qquad n_i \in \mathbb{Z}.$$

Let G be some abelian group, such as $\mathbb{Z}_n, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

Definition: The chain complex $C_*(X;G)$ is defined as follows: the n^{th} chain group

$$C_n(X;G) = \big\{ \sum_i n_i \sigma_i \mid \sigma_i \in C_n(X), n_i \in G \big\}.$$

The sum $\sum_i n_i \sigma_i$ is finite. The boundary operators are defined analogously to the boundary operators for the chain complex $C_*(X)$.

One can describe this in terms of tensor products as follows:

$$C_n(X;G) \approx C_n(X) \otimes G.$$

The boundary operators are then just $\partial \otimes$ id. Homology with coefficients in G is then defined by

$$H_*(X;G) = H_*(C_*(X;G)).$$

Essentially everything (excision, Mayer-Vietoris, etc.) goes through verbatim in this setting. However, one thing that does change is the following:

The dimension axiom: $H_0(pt, G) \approx G$.

Example: $\mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n$. The cellular chain complex is:

$$\begin{array}{ccccccccccccccc}
\cdots & \xrightarrow{\partial} & C_{k+1} & \xrightarrow{\partial} & C_k & \xrightarrow{\partial} & \cdots & \xrightarrow{\partial} & C_2 & \xrightarrow{\partial} & C_1 & \xrightarrow{\partial} & C_0 & \to 0 \\
 & & || & & || & & & & || & & || & & || & \\
\cdots & \to & \mathbb{Z} & \to & \mathbb{Z} & \to & \cdots & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \to 0
\end{array}$$

With $\mathbb{Z}_2$ coefficients, $C_* \otimes Z_2 = C_*(\mathbb{R}P^n; Z_2)$, the chain groups become either 0 or $\mathbb{Z}_2$, and all boundary maps become 0:

$$\cdots \xrightarrow{0} 0 \to \mathbb{Z}_2 \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} 0$$

Therefore,

$$H_i(\mathbb{R}P^n; \mathbb{Z}_2) = \begin{cases} Z_2 & i = 0, 1, \ldots, n \\ 0 & \text{otherwise.} \end{cases}$$

5.3 Application: Lefschetz Fixed Point Theorem

Definition: Let A be a finitely generated abelian group and let $\phi : A \to A$ be a group homomorphism. Then $A = F \oplus T$, where F is free abelian of rank $r < \infty$ and T is a finite group. ϕ induces a group homomorphism $\phi : A/T \to A/T$, which can be represented by an $n \times n$ matrix over $\mathbb{Z}$.. The *trace* of ϕ is the trace of this $n \times n$ matrix (the sum of the diagonal entries). The choice of basis is not important, since the trace is invariant under conjugation. The trace of ϕ also equals the trace of the induced linear transformation on the vector space $A \otimes \mathbb{Q}$.

Definition: Suppose X is a finite complex, and $f : X \to X$ is a map. Then f induces a homomorphism $f_* : H_n(X) \to H_n(X)$ of finitely generated abelian groups. The *Lefschetz Number*

$$\lambda(f) \stackrel{def}{=} \sum_n (-1)^n \text{trace}(f_*(H_n(X; \mathbb{Q}) \to (H_n(X; \mathbb{Q})).$$

The reason for taking coefficients in $\mathbb{Q}$ is to kill off the torsion part of $H_*(X)$..

Theorem: (Lefschetz) If f is fixed point free, then $\lambda(f) = 0$.

Example: If $f \simeq \text{id}$, then we are computing the traces of identity matrices, and $\lambda(f) = \chi(X)$, the Euler characteristic of the space X.

Let β_n be the rank of the finite dimensional vector space $H_n(X;\mathbb{Q})$. β_n is called the nth Betti number. Then the Euler characteristic of X was defined to be

$$\chi(X) = \sum_n (-1)^n \beta_n.$$

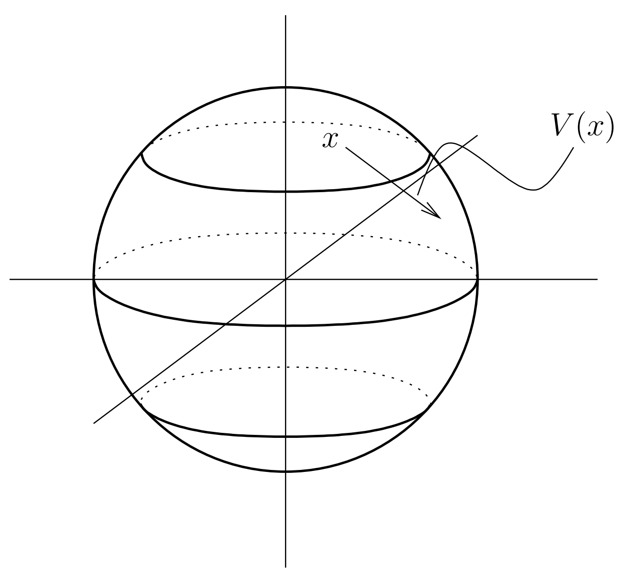
Let γ_n be the the number of n-dimensional cells in X.

Theorem: (essentially using only algebra) $\chi(X) = \sum(-1)^n \gamma_n$.

Example: Let $X = S^{2n}$. Then $\chi(X) = 2$, since

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0, 2n \\ 0 & \text{otherwise} \end{cases}$$

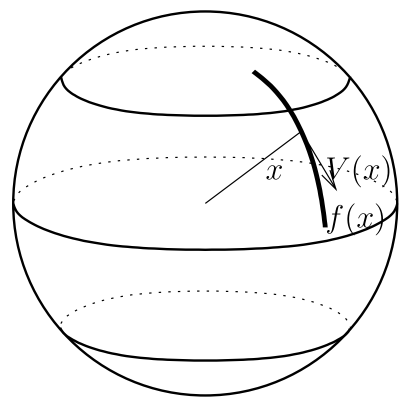
Suppose that X has a nonvanishing vector field V



So $x \in \mathbb{R}^{2n+1}$ is a vector with norm 1 and $V(x) \in \mathbb{R}^{2n+1}$ is perpendicular to x ($x \cdot V(x) = 0$).

Now $V : x \to V(X)$ is continuous in x, $V(x) \neq 0$ for all x, and $x \cdot V(x) = 0$ for all x. Without loss of generality, assume that $|V(x)| = 1$ for all x.

There exists a unique geodesic on S^{2n} going through x in the direction of $V(x)$. Let $f : S^{2n} \to S^{2n}$ be the map that takes x to that point $f(x)$ whose distance from x along this geodesic is 1.



f is clearly homotopic to the identity map. Therefore

$$\lambda(f) = \lambda(id) = \chi(S^{2n}) = 2.$$

But f has no fixed points and therefore $\lambda(f) = 0$. This is a contradiction. Thus S^{2n} does not have a non-vanishing vector field.

We can generalize this example to any smooth manifold.

Theorem: If M^n is a compact n-manifold with non-zero Euler characteristic, then M does not have a nonzero vector field.

Example $\mathbb{C}P^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ has Euler characteristic $\chi(\mathbb{C}P^n) = n+1$. therefore, $\mathbb{C}P^n$ does not have a nonzero vector field.

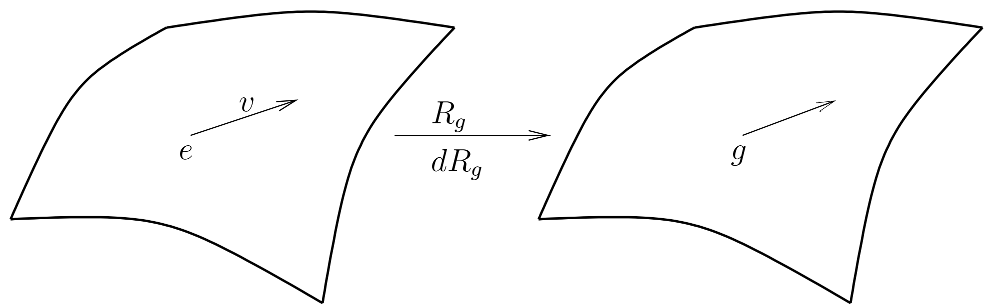
Example: Any odd dimensional sphere, S^{2n+1}, has a nonzero vector field. Let $x = (x_1, \ldots, x_{2n+2}) \in S^{2n+1}, \sum x_i^2 = 1$. Then

$$V(x) = (-x_2, x_1, -x_4, x_3, \cdots, -x_{2n+2}, x_{2n+1})$$

defines such a vector field.

Example: Let G be a topological group which is also a smooth manifold. For example: S^1, S^3 or $SO(n) = n \times n$ orthogonal matrices. Then the Euler characteristic of G is zero. To see this we construct a nonvanishing vector field.

Let $e \in G$ be the identity element. Take $v \in T_e(G), v \neq 0$. Then we construct a nonzero vector field V by $V(g) = (dR_g)(v)$ where $R_g : G \to G$ is right multiplication by g, and d is differentiation.



Chapter 6

January 20

6.1 Tensor Products

Suppose A, B are abelian groups. We define their tensor product as follows:

Definition: $A \otimes B = F(A \times B)/N$, where $F(A \times B)$ is the free abelian group on pairs $(a, b) \in A \times B$ and N is the subgroup which is generated by all elements of the form $(a_1+a_2, b)-(a_1, b)-(a_2, b)$ or $(a, b_1+b_2)-(a, b_1)-(a, b_2)$.

Thus we are forcing linear relations in both coordinates. **Notation:** $a \otimes b$ denotes the class of (a, b) in $A \otimes B$. $A \otimes B$ is an abelian group.

Zero element: $a \otimes 0 = 0 \otimes a = 0$

Inverses: $-(a \otimes b) = (-a) \otimes b = a \otimes (-b)$.

Properties:

- $A \otimes \mathbb{Z} \approx A$. The isomorphism is given by $a \otimes n \to na$, and the inverse isomorphism is $A \to A \otimes \mathbb{Z}$, $a \to a \otimes 1$.
- $A \otimes \mathbb{Z}_n \approx A/nA$. The homomorphism $\phi : A \to A \otimes \mathbb{Z}_n, \phi(a) = a \otimes 1$

induces an isomorphism $A/nA \to A \otimes Z_n$. To see this note that $\phi(na) = na \otimes 1 = a \otimes n = 0$.

- $A \otimes B \approx B \otimes A$.
- $\otimes$ is associative, that is $(A \otimes B) \otimes C \approx A \otimes (B \otimes C)$.
- Suppose A is finite. Then $A \otimes \mathbb{Q} = 0$ since
$$a \otimes r/s = ra \otimes 1/s = nra \otimes 1/ns = 0 \text{ for some } n$$
.
- $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C)$
- $\mathbb{Z}_m \otimes Z_n \approx \mathbb{Z}_d$, where $d = \gcd(m, n)$.
- $A \otimes B$ is functorial in both A and B. For example, if $\phi : A \to A'$ is a group homomorphism, then there exists a group homomorphism $\phi \otimes 1 : A \otimes B \to A' \otimes B$.

Question: What is the relationship between $H_n(X; G)$ and $H_n(X) \otimes G$? Recall that $C_*(X; G) = C_*(X) \otimes G$, and $H_*(X; G) = H_*(C_*(X) \otimes G)$.

Take $c \in C_n(X)$, and suppose that c is a cycle. Let $[c]$ be the homology class, $[c] \in H_n(X)$. Now, $c \otimes g \in C_n(X) \otimes G$ is a cycle, since $\partial(c \otimes g) = \partial(c) \otimes g$. Therefore, $[c \otimes g]$ is a homology class in $H_n(X; G)$.

Definition: $\mu : H_n(x) \otimes G \to H_n(X; G)$ is the map $\mu : [c] \otimes g \to [c \otimes g]$

μ is not an isomorphism, in general. However, we have the following theorem:

Theorem: (Universal Coefficient theorem for Homology): There exists a natural short exact sequence

$$0 \longrightarrow H_n(X) \otimes G \xrightarrow{\mu} H_n(X; G) \longrightarrow \mathrm{Tor}(H_{n-1}(X), G) \longrightarrow 0$$

The definition of the Tor functor and the proof will be given later.

6.2 Lefschetz Fixed Point Theorem

Definition: Suppose $\phi : A \to A$ is an endomorphism of the finitely generated abelian group A. By the fundamental theorem of abelian group theory $A \approx T \oplus F$, where T is a finite group and F is a free abelian group of finite rank. Then ϕ induces an endomorphism $\bar{\phi} : F \to F$. The trace of ϕ is defined by $\mathrm{Tr}\phi = \mathrm{Tr}\bar{\phi}$.

Lemma: Suppose we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & A'' & \longrightarrow & 0 \\ \downarrow & & \phi\downarrow & & \phi'\downarrow & & \phi''\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & A'' & \longrightarrow & 0 \end{array}$$

where A, A', A'' are finitely generated abelian groups. Then

$$\mathrm{Tr}\phi' = \mathrm{Tr}\phi + \mathrm{Tr}\phi''$$

Lemma: (Hopf Trace Formula) Suppose C_* is a chain complex of abelian groups such that:

- Each C_n is a finitely generated abelian group.
- $C_n \neq 0$ for only finitely many n.

Suppose that $\phi : C_* \to C_*$ is a chain map. Then

$$\sum_n (-1)^n \mathrm{Tr}(\phi : C_n \to C_n) = \sum_n (-1)^n \mathrm{Tr}(\phi_* : H_n(C) \to H_n(C))$$

Proof:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

Let $Z_n := \ker \partial_n$, $B_n := \partial_{n+1}(C_{n+1})$, so $H_n := Z_n/B_n$.

Consider

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d, "\phi"'] & Z_n \arrow[r] \arrow[d, "\phi'"'] & H_n \arrow[r] \arrow[d, "\phi_*"'] & 0 \\
0 \arrow[r] & B_n \arrow[r] & Z_n \arrow[r] & H_n \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & Z_n \arrow[r] \arrow[d, "\phi'"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "\phi"'] & B_{n-1} \arrow[r] \arrow[d, "\phi"'] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r, "\partial_n"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

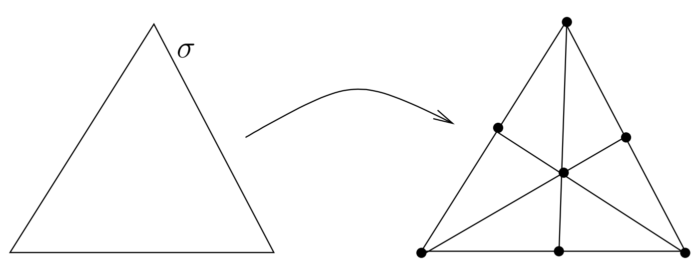
Now,

$$\mathrm{Tr}(\phi_* : H_n \to H_n) = \mathrm{Tr}(\phi' : Z_n \to Z_n) - \mathrm{Tr}(\phi'' : B_n \to B_n)$$
$$\mathrm{Tr}(\phi : C_n \to C_n) = \mathrm{Tr}(\phi' : Z_n \to Z_n) + \mathrm{Tr}(\phi'' : B_{n-1} \to B_{n-1})$$

Therefore $\sum_n (-1)^n \mathrm{Tr}(\phi_* : H_n \to H_n) = \sum_n (-1)^n \mathrm{Tr}(\phi : C_n \to C_n)$ □

Proof: of the Lefschetz fixed point formula: For the sake of simplicity assume that X is a finite simplicial complex and f is a simplicial map. Suppose $f(x) \neq x$ for all $x \in X$.

It is **not necessarily true** that $f(\sigma) \cap \sigma =$ for all simplices σ in X, but it is true if we asubdivide sufficiently many times.



Repeat until the simplices are small enough so that $f(\sigma) \cap \sigma = \emptyset$. We can do this due to compactnes.

Then $\sum (-1)^n \mathrm{Tr}(f : C_n \to C_n) = 0$ and so $\lambda = 0$. □

Example: $f : \mathbb{R}P^{2k+1} \to \mathbb{R}P^{2k+1}$, $\lambda(f) = 1 - \deg f$.

$$\begin{array}{ccc} H_{2k+1}(\mathbb{R}P^{2k+1}) & \xrightarrow{f_*} & H_{2k+1}(\mathbb{R}P^{2k+1}) \\ \approx\downarrow & & \approx\downarrow \\ \mathbb{Z} & \xrightarrow{\times \deg f} & \mathbb{Z} \end{array}$$

If f is fixed point free, then $\deg f = 1$ (e.g. a homoemorphism that preserves orientation).

$$f(x_1, x_2, \ldots, x_{2k+1}, x_{2k+2}) = (x_2, -x_1, \ldots, x_{2k+2}, -x_{2k+1})$$

f is a homeomorphism. Moreover, f preserves orientation, because as a linear map, f has matrix

$$\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}$$

6.3 Cohomology

Start with a chain complex $C_*(x)$.

$$0 \leftarrow C_0 \overset{\partial_1}{\leftarrow} C_1 \overset{\partial_2}{\leftarrow} C_2 \overset{\partial_3}{\leftarrow} \cdots \overset{\partial_n}{\leftarrow} C_n \overset{\partial_{n+1}}{\leftarrow} C_{n+1} \leftarrow \cdots$$

and dualize to get the *cochain complex*: $C^* := Hom(C_*(X), G)$

$$0 \to C^0 \overset{\delta^0}{\to} C^1 \overset{\delta^1}{\to} C^2 \overset{\delta^3}{\to} \cdots \overset{\delta^{n-1}}{\to} C^n \overset{\delta^n}{\to} C^{n+1} \to \cdots$$

where $\delta^n : \mathrm{Hom}(C_n(X), G) \to \mathrm{Hom}(C_{n+1}(X), G)$ maps $\alpha : C_n(X) \to G$ to $\alpha \circ \delta^{n+1}$:

$$\delta^n(\alpha) : C_{n+1} \overset{\partial_{n+1}}{\to} C_n \overset{\alpha}{\to} G$$

$H^*(X; G)$ =the homology of this chain complex.

Cohomology is contravariant – i.e. if $f : X \to Y$, $f^* : H^n(Y) \to H^n(X)$.

Question: What is the relationship between $H^*(X;G)$ and $\mathrm{Hom}(H_n(X),G)$?

Cohomology has more structure than homology. There exists a product

$$H^p(X) \times H^q(X) \to H^{p+q}(X \times X).$$

Let $d : X \to X \times X$ be the diagonal: $d(x) = (x,x)$. Thereforeore, there exists a product, called the *cup product*:

$$\begin{gathered} H^p(X) \otimes H^q(X) \to H^{p+q}(X) \\ u \otimes v \mapsto u \cup v \end{gathered}$$

$H^*(X)$ is a graded ring.

Examples:

- $H^*(\mathbb{C}P^n;\mathbb{Z})$ is a polynomial algebra; it is isomorphic to $Z[x]/(x^{n+1})$, $x \in H^2(\mathbb{C}P^n;Z)$.
- $H^*(\mathbb{R}P^n;\mathbb{Z}_2) \approx \mathbb{Z}_2[x]/(x^{n+1})$, $x \in H^1(\mathbb{R}P^n;\mathbb{Z}_2) \approx \mathbb{Z}_2$; $x^2 = x \cup x \in H^2$.

Chapter 7

January 25

7.1 Examples: Lefschetz Fixed Point Formula

Basic notion: A is a finitely generated abelian group, and $\phi : A \to A$ is an endomorphism. The fundamental theorem of finitely generated abelian groups says that

$$A \approx F \oplus T; \qquad F \approx \mathbb{Z}^r, T \text{ finite group.}$$

To get a trace for ϕ we do one of the following:

- Take the trace of the linear transformation $\phi \otimes 1$:

$$\begin{tikzcd}
A \otimes Q \arrow[r, "\phi \otimes 1"] \arrow[d, "\approx"] & A \otimes Q \arrow[d, "\approx"] \\
Q^r \arrow[r, "\text{linear}"'] & Q^r
\end{tikzcd}$$

- Take the trace of $\bar{\phi}$:

$$\begin{tikzcd}
A/T \arrow[r, "\bar{\phi}"] \arrow[d, "\approx"] & A/T \arrow[d, "\approx"] \\
\mathbb{Z}^r \arrow[r, "\bar{\phi}"] & \mathbb{Z}^r
\end{tikzcd}$$

Lefschetz Fixed Point Theorem. If X is finite and $f : X \to X$ is a map wiht no fixed points, then $\lambda(f) = 0$. Recall:

$$\lambda(f) = \sum -1^n \mathrm{Tr}(f_* : H_n(X;\mathbb{Q}) \to H_n(X;\mathbb{Q}))$$

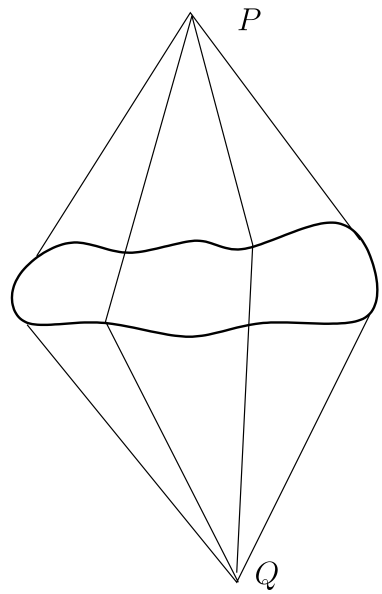
Question. Is the converse of the Lefschetz Fixed Point Theorem true?

Answer: **NO**.

Remark: The Lefschetz Fixed Point Theorem is true for coeffecients in any field F.

Example. $f : X \to X$, for a finite complex X. Consider the suspension:

$$Sf : SX \to SX.$$



Note that Sf has two fixed points: P and Q. Similarily, the k-fold suspension $S^k f : S^k X \to S^k X$ has a sphere of fixed points.

Recall:

$$\begin{tikzcd}
\tilde{H}_i(X) \arrow[r, "\approx"] \arrow[d, "f_*"] & \tilde{H}_{i+k}(X) \arrow[d, "(S^k f)_*"] \\
\tilde{H}_i(X) \arrow[r, "\approx"] & \tilde{H}_{i+k}(X)
\end{tikzcd}$$

Therefore, we can relate $\lambda(f)$ to $\lambda(S^k f)$:

i	0	1	2	...	k	$k+1$	...
$H_i(X)$	$\mathbb{Z}$	H_1	H_2	...	H_k	H_{k+1}	...
$H_i(S^k X)$	$\mathbb{Z}$	0	0	0	0	H_1	...

Assume that X is connected. Then $\lambda(f)-1=(-1)^k(\lambda(S^kf)-1)$ ($f_*: H_0(X)\to H_0(X)$ is identity on $\mathbb{Z}$)

The converse of the Lefschetz Fixed Point Theorem would say that $\lambda(g)=0 \overset{?}{\Rightarrow} g$ has no fixed points. We shall choos f and k such that $\lambda(S^kf)=0$ but S^kf has lots of fixed points.

For example, start with a map $f: X\to X$ such that $\lambda(f)=0$ and assume that k is even. Then $\lambda(S^kf)=0$.

This suggests the question: let $f: X\to X$, where X is a finite complex. What are the possible sequences of integeres $k_0,k_1,k_2,\ldots$ that realize the traces $f_*: H_i\to H_i$? Assume that X is connected, so that $l_0=1$. Recall: there exists a map of degree l, $\phi: S^1\to S^1$ such that $z\mapsto z^l$, where $z\in \mathbb{C}, |z|=1$. This map has a fixed point (the complex number 1) so it is base point preserving.

The $(k-1)$fold suspension of ϕ, $S^k\phi: S^k\to S^k$ has degree l, and it is also base point preserving.

Suppose that $X=S^{n_1}\vee X=S^{n_2}\vee\cdots\vee X=S^{n_r}$ where $n_1\leq n_2\leq\cdots\leq n_r$. On each sphere S^{n_k}, choose a self map of degree k_j. Therefore, we can realize any sequence $(1,k_1,k_2,\ldots)$ by choosing a one point union of spheres. This is a kind of "cheap" example, since all of the attaching maps are trivial.

Now choose

$$\begin{aligned} X &= S^2\vee S^4\vee S^6\vee\cdots\vee S^{2n}Y && =\mathbb{C}P^n \\ &= e^2\vee e^4\vee e^6\vee\cdots\vee e^{2n} \end{aligned}$$

Here, $H_*(X,\mathbb{Z})\approx H_*(Y;Q)$. We can realize any sequence $(1,k_1,k_2,\ldots,k_n)$ for traces of self maps $f: X\to X$. This is not the case in Y.

i	0	1	2	3	4	$\ldots$	$2n-1$	$2n$
$H_i(Y)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\ldots$	0	$\mathbb{Z}$
$\mathrm{Tr} f_*$	1	0	k_1	0	k_2		0	k_n

The values of k are determined completely by k_1: $k_2=k_1^2$, $k_3=k_1^3$, and so forth. To prove this, we need to use cohomology: we need a ring structure.

7.2 Applying Cohomology

For any space W and a ring R, we can put a natural graded ring structure on the cohomology groups $H^*(W;R)$. That is, if $u \in H^p(W;R)$ and $v \in H^q(W;R)$, the their product, $u \cup v \in H^{p+q}(W;R)$. If $f : W \to Z$ is a map, then the (contravariantly) induced map

$$f^* : H^*(Z,R) \to H^*(W;R)$$

is a ring homomorphism, then there exists an identity $1 \in H^0(X;R)$.

Fact: $H^*(\mathbb{C}P^n;\mathbb{Z}) \approx Z[x]/(x^{n+1})$, where $x \in H^2(CP^n;\mathbb{Z}) \approx \mathbb{Z}$ is a generator. Take any self map $f : Z \to X$. Then we can compute the traces using cohomology. Consider the map $f*$:

$$\begin{tikzcd}
H^2(\mathbb{C}P^n) \arrow[r, "f^*"] \arrow[d, "\approx"] & H^2(\mathbb{C}P^n) \arrow[d, "\approx"] \\
\mathbb{Z} \arrow[r] & \mathbb{Z} \\
x \arrow[r, mapsto] & kx
\end{tikzcd}$$

Here, $f^*(x^l) = f^*(x)^l = (kx)^l = k^l x^l$, since f^* is a ring homomorphism. Therefore, the sequence of trances is $(1, k, k^2, \ldots, k^n)$. Moreover, we can realize any k.

7.3 History: The Hopf Invariant 1 Problem

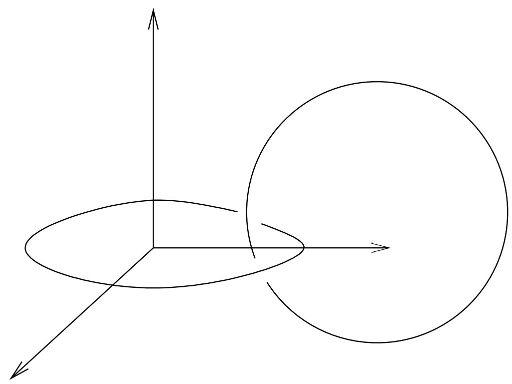
Suppose $f : S^3 \to S^2$. Then $f_* : \tilde{H}_*(S^3) \xrightarrow{0} \tilde{H}_*(S^2)$. Question: Does f induce the 0 map in homotopy? Here, $\Pi_n(X) :=$ the group of homotopy classes of maps $S^n \to X$. $f : S^3 \to S^2$ induces $f_\# : \Pi_n(S^3) \to \Pi_n(S^2), [g] \mapsto [f \circ g]$.

In the 1930s, Heinz Hopf constructed the following map $f : S^3 \to S^2$:

$$(z_1, z_2) \mapsto \frac{z_1}{z_2}$$

where $z_1, z_2 \in \mathbb{C}, z_1\bar{z}_1 + z_2\bar{z}_2 = 1$ and the image lies in the one point compactification $\mathbb{C} \cup \infty$ of $\mathbb{C}$.

Pick $x, y \in S^2, x \neq y$. Then $f^{-1}(x), f^{-1}(y)$ are both circles.



Facts:

1. $f_\#$ is an isomorphism for $n > 3$.

2. $\pi_3 S^2 \approx Z$.

Let $f : S^{2n-1} \to S^n$ $(n \geq 2)$. Then $X = S^n \cup_f e^2 n$. Then

$$H_i(X) \approx \begin{cases} Z & i = 0, n, 2n \\ 0 & \text{otherwise} \end{cases}$$

Let $y \in H^{2n}(X) \approx \mathbb{Z}$ be a generator. Then $x^2 = x \cup x = ky$. What is k? In particular, when is there a map $f : S^{2n-1} \to S^n$ with $k = 1$? k is called the Hopf invariant of f

7.4 Axiomatic Description of Cohomology

Definition: A *cohomology theory* is a sequence of contravariant functors from the category of CW complexes to the category of abelian groups (say $\tilde{h}^n, n \in \mathbb{Z}$) together with *natural* transformations $\tilde{h}^n(A) \xrightarrow{\partial} \tilde{h}^{n+1}(X/Z)$ for every CW pair (X, A) satisfying the following axioms:

1. (homotopy axiom) If $f \simeq g : X \to y$ then $f^* = g^* : \tilde{h}^n(Y) \to \tilde{h}^n(X)$.
2. The following is a long exact sequence:

   $$\ldots \longrightarrow \tilde{h}^n(X/A) \xrightarrow{c^*} \tilde{h}^n(X) \xrightarrow{i^*} \tilde{h}^n(A) \xrightarrow{\partial} \tilde{h}^{n+1}(X/A) \ldots$$

   The transformations ∂ are *natural*. This means that if $f : (X, A) \to (Y, B)$ is a map of CW complexes, then the following diagram is commutative:

$$\begin{tikzcd}
\ldots \arrow[r] & \tilde{h}^n(X/A) \arrow[r, "c^*"] & \tilde{h}^n(X) \arrow[r, "i^*"] & \tilde{h}^n(A) \arrow[r, "\partial"] & \tilde{h}^{n+1}(X/A) \arrow[r] & \ldots \\
\ldots \arrow[r] & \tilde{h}^n(Y/B) \arrow[r, "c^*"] \arrow[u, "f^*"'] & \tilde{h}^n(Y) \arrow[r, "i^*"] \arrow[u, "f^*"'] & \tilde{h}^n(B) \arrow[r, "\partial"] \arrow[u, "f^*"'] & \tilde{h}^{n+1}(Y/B) \arrow[r] \arrow[u, "f^*"'] & \ldots
\end{tikzcd}$$

3. Suppose $X = \bigvee_{\alpha \in A} X_\alpha$, where A is the same index set. Let $i_\alpha : X_\alpha \to X$ be the inclusion maps. Therefore, $i_\alpha^* : \tilde{h}^n(X) \to \tilde{h}^n(X_\alpha)$, $\alpha \in A$. These maps induce an isomorphism $\tilde{h}^n(X) \xrightarrow{\approx} \prod_{\alpha \in A} \tilde{h}^n(X_\alpha)$.

The first axiomatization of cohomology was done by Eilenberg and Steenrod, in the book “Algebraic Topology”.

7.5 My Project

Start to read Milnor’s book: Characteristic Classes.

Chapter 8

January 27

8.1 A Difference Between Homology and Cohomology

Example. Let G be a free abelian groups on generators e_α, $\alpha \in A$.

Elements of G: $\sum_{a \in A} n_\alpha e_\alpha$, where $n_\alpha \in \mathbb{Z}$i and the n_α are finitely nonzero.

Let $G \approx \oplus_{a \in A} \mathbb{Z}_\alpha$. Then $G^* = \text{Hom}\,(G, \mathbb{Z})$. $f : G \to \mathbb{Z}$, $x_\alpha = f(e_\alpha)$, for $\alpha \in A$.

Hom $(G, \mathbb{Z}) = \prod_{\alpha \in A} \mathbb{Z}_\alpha$. A typical element in $\pi_{\alpha \in A} \mathbb{Z}_\alpha$ is a sequence $(x_\alpha)_{\alpha \in A}$, not necessarily finitely nonzero.

8.2 Axioms for Unreduced Cohomology

Last time we gave axioms for reduced cohomology: a sequence of contravariant functors $\tilde{h}^n$.

Unreduced cohomology: h^n contravariant functors.

$$\begin{aligned} h^n(X, A) &= \tilde{h}^n(X/A) \\ h^n(X) &= h^n(X, \emptyset) \\ &= \tilde{h}^n(X/\emptyset) \end{aligned}$$

where $X/\emptyset = X^+ = X \uplus$ point.

Then there are axioms for the unreduced cohomology theory. See text.

8.3 Eilenberg-Steenrod Axioms

Eilenberg-Steenrod (c. 1950) in Foundations of Algebraic Topology, gave the first axiomatic treatment, including the following:

Dimension Axiom: $h^n(pt) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{otherwise} \end{cases}$.

The above axioms and the dimension axiom give a unique theory.

Remark: there exist homology (cohomology) theories which do not satisfy the dimension axiom. (for a trivial example, homology with coefficients)

An uninteresting example: Suppose h^* is a cohomology theory satisfying the dimension axiom. Define another cohomology theory by $k^n = h^{n-m}$ for a fixed integer m.

1960s: Atiyah, Bott, Hirzebruch constructed another cohomology theory (complex K-theory) which does not satisfy the dimension axiom:

$$K^n(pt) = \begin{cases} \mathbb{Z} & n = 0, \pm 2, \pm 4, \ldots \\ 0 & \text{otherwise} \end{cases}$$

This is a very powerful theory and is part of the reason that Atiyah got a fields medal.

Definition: The *coefficients* of a cohomology theory are $h^*(pt)$.

Another example of a sophisticated theory is stable cohomotopy theory; the groups are stable cohomotopy groups of spheres.

8.4 Construction of a Cohomology Theory

Remark: We can develop cohomology from the axioms. But we need a construction of a cohomology theory.

Start with a chain complex C_*.

Definition: the *cochain group* $C^* :=$ Hom (C_*, G) where G is an abelian group, called the *coefficients*.

A typical element in $C^n :=$ Hom (C_n, G) is a group homomorphism $\alpha : C_n \to G$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dr, "\alpha \circ \partial_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "\alpha"] & C_{n-1} \arrow[r] & \cdots \\
 & & G & &
\end{tikzcd}$$

Therefore, there exists a coboundary map $\delta^n : C^n \to C^{n+1}$, $\delta^n(\alpha) := \alpha \circ \partial_{n+1}$.

It is an elementary fact that $\delta \circ \delta = 0$.

Definition: $H^n(C; G) = \ker \delta / \text{image} \delta$.

Example: $H^*(X; G) = H^*(\text{Hom }(C_*(X), G)))$

We can do it for a pair also:

$$H^*(X, A; G) := H^*\left(\text{Hom }\left(\frac{C(X)}{C(A)}, G\right)\right)$$

Then the axioms follow algebraically (although there is some work to be done in order to check that this is so!)

Example: Long exact sequence in homology. Suppose E is the short exact sequence of abelian groups.

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

Let G be an abelian group. Then there exists an exact sequence

$$\text{Hom }(A, G) \xleftarrow{i^*} \text{Hom }(B, G) \xleftarrow{j^*} \text{Hom }(C, G) \longleftarrow 0$$

with $i^*(\beta) = \beta \circ i$ for a homomorphism $\beta : B \to G$.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[dr, "i^*(\beta) = \beta \circ i"'] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r] & 0 \\
& & G & &
\end{tikzcd}$$

Moreover, if E is a split exact seqnece, then $i^* : \text{Hom }(B, G) \to \text{Hom }(A, G)$ is an epimorphism, i.e.

$$0 \longleftarrow \text{Hom }(A, G) \xleftarrow{i^*} \text{Hom }(B, G) \xleftarrow{j^*} \text{Hom }(C, G) \longleftarrow 0$$

is exact.

Proof: Exactness at Hom (C, G) : j^* is 1-1: Let $\gamma : C \to G$ be in $\ker j^*$, i.e. $\gamma \circ j = 0$. But $j : B \to C$ is an epimorphism, so $\gamma = 0$.

The following is clear: $\mathrm{im} j^* \subseteq \ker i^*$.

Next, we show that $\ker i^* \subseteq \mathrm{im}\ j^*$. Take an element $\beta \in \ker i^*$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] \arrow[dr, "\beta\circ i = 0"'] & B \arrow[r, "j"] \arrow[d, "\beta"] & C \arrow[r] \arrow[dl, "\gamma"] & 0 \\
& & G & &
\end{tikzcd}$$

Define $\gamma : C \to G$ by $\gamma(c) = \beta(b)$, where b is any element such that $j(b) = c$.

Now assume that E is split exact:

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ j\ } C \longrightarrow 0$$

That is, there exists $p : C \to B$ such that $j \circ p = \mathrm{id}_C$, or equivalently, there exists $q : B \to A$ such that $q \circ i = \mathrm{id}_A$. Exercise: In this case, $B \approx A \oplus C$.

Then

$$0 \longleftarrow \mathrm{Hom}\ (A, G) \xleftarrow{\ i^*\ } \mathrm{Hom}\ (B, G)$$

is exact:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, shift right, "i"'] \arrow[d, "\alpha"'] & B \arrow[l, shift right, "q"'] \arrow[r] \arrow[dl, "\beta"] & \ldots \\
& G & &
\end{tikzcd}$$

Comments.

1. If C is a free abelian group, then E is split exact.

2. Consider the short exact sequence of chain complexes:

$$0 \longrightarrow C_*(A) \longrightarrow C_*(X) \longrightarrow C_*(C, A) \longrightarrow 0$$

These chain groups are free, and therefore the sequence is split. Therefore, we have a short exact sequence of cochain complexes,

$$\begin{array}{ccccccccc} 0 & \leftarrow & C^*(A) & \leftarrow & C^*(X) & \leftarrow & C^*(X,A) & \leftarrow & 0 \\ & & \Vert & & \Vert & & \Vert & & \\ 0 & \leftarrow & \text{Hom }(C_*(A), G) & \leftarrow & \text{Hom }(C_*(X), G) & \leftarrow & \text{Hom }(C_*(X,A), G) & \leftarrow & 0 \end{array}$$

By general nonsense, this gives the long exact sequencne in cohomology.

8.5 Universal Coefficient Theorem in Cohomology

Question: what is the relationship between $H^*(X, G)$ and Hom $(H_*(X), G)$? In the first group, we dualize the chain complex and then apply homology; in the second, we apply homology to the chain complex first, and then dualize.

The two groups are not isomorphic; however, There exists a group homomorphism

$$H^*(X; G) \xrightarrow{h} \text{Hom }(H_*(X), G)$$

which is onto.

Definition: of h: let $h \in H^n(X;G)$ be a cohomology class, represented by $\alpha : C_n(X) \to G$:

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1}(X) \arrow[r] \arrow[dr, "\alpha \circ \delta_{n+1} = 0"'] & C_n(X) \arrow[r] \arrow[d, "\alpha"] & C_{n-1}(X) \arrow[r] \arrow[dl, "\beta"] & \dots \\
& & G & &
\end{tikzcd}$$

$h(u)$ is to be a homomorphism $H_n(X) \to G$. α is not unique, $\alpha + \beta\partial_n$ will also do.

Restrict α to $\alpha' : \ker \partial_n = Z_n(X) \to G$.

But $\alpha'(B_n) = 0$ and therefore there exists a homomorphism $H_n(X) \to G$. Define $h(u)$ to be this homomorphism.

Exercise: h is onto.

h is not always an isomorphism. Example:

$$0 \longrightarrow \ker h \longrightarrow H^*(\mathbb{R}P^n) \xrightarrow{h} \text{Hom }(H_*(\mathbb{R}P^n), \mathbb{Z}) \longrightarrow 0$$

We will compute $H_*(\mathbb{R}P^n)$ cellularly:

$$\begin{tikzcd}
\dots \arrow[r] & C_3 \arrow[r] \arrow[d, equal] & C_2 \arrow[r] \arrow[d, equal] & C_1 \arrow[r] \arrow[d, equal] & C_0 \arrow[r] \arrow[d, equal] & 0 \\
\dots \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} &
\end{tikzcd}$$

and therefore

$$H_i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}_2 & \text{if } i \text{ is odd}; 1 \leq i \leq n \\ 0 & \text{if } i \text{ is even}; 2 \leq i \leq n \\ 0 & \text{if } i = n \text{ is even} \\ \mathbb{Z} & \text{if } i = n \text{ is odd} \end{cases}$$

Dualize the chain complex C_* to get $C^* = \text{Hom}\,(C_*, \mathbb{Z})$:

$$\begin{array}{ccccccccccccc} 0 & \leftarrow & C^n & \leftarrow & \cdots & \leftarrow & C^2 & \leftarrow & C^1 & \leftarrow & C^0 & \leftarrow & 0 \\ & & \| & & & & \| & & \| & & \| & & \\ & & \mathbb{Z} & & & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \end{array}$$

Therefore,

$$H^i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i \text{ odd, } 1 \leq i \leq n \\ \mathbb{Z}_2 & \text{if } i \text{ even, } 1 \leq i \leq n \end{cases}$$

Note that these are very different from the homology groups! So,

$$H^i(\mathbb{R}P^n) \xrightarrow{h} \text{Hom}\,(H_i(\mathbb{R}P^n), \mathbb{Z}) \longrightarrow 0$$

is the zero map for $0 \leq i \leq n$. Therefore $H^i(\mathbb{R}P^n) \approx \ker h$ in these dimensions. $\square$

Chapter 9

February 1

9.1 Comments on the Assignment

.

#2 p.184:

Use the Lefschetz Fixed Point Theorem to show that a map $f : \S^n \to \S^n$ has a fixed point unless $\deg f$ = the degree of the antipodal map. Fairly elementary: just look at what the LFP number for f is $\lambda(f) = 1 + (-1)^n \deg f$, and the theorem relates this to fixed points.

#4 p.184: Suppose that X is a finite simplicial complex, $f : X \to X$ is a simplicial homeo. Let $F = \{x \in X | f(x) = x)$. Show that $\lambda(f) = X(F)$. We may assume that F is a subcomplex of X (if not, one can subdivide).

#13 p.206: $\langle X, Y\rangle$ is the set of base point preserving homotopy classes of maps $f : X \to Y$. We are to show that there exists an isomorphism $\langle X, K(G, 1)\rangle \to H^1(X, G)$, where G is some abelian group. A space W is a $K(G, 1)$ if $\pi_1(W) = G$ and the universal covering space $\tilde{W}$ is contractible ($\Longleftrightarrow \pi_n(\tilde{W}) = 1$ for all $n \geq 2$.)

The map $\langle X, K(G,1)\rangle \to H^1(X;G)$ is defined as follows: $f : X \to K(G,1)$ induces $f_* : H_1(X) \to H_1(K(G,1)) \approx G$. Note that $H_1(Y) \approx \pi_1(Y)/[\pi_1(Y), \pi_1(Y)]$ for any space Y. Since G is abelian this gives us a map

$$\langle X, K(G,1)\rangle \to \text{Hom }(H_1(X), G) \approx H^1(X;G)$$

The isomorphism Hom $(H_1(X), G) \approx H^1(X;G)$ comes from the UCT in cohomology, which says that $\exists$ a short exact sequence:

$$0 \to \ker h \to H^n(Y;G) \xrightarrow{h} \text{Hom }(H_n(Y), G) \to 0.$$

Moreover, $\ker h = 0$ if $n = 1$.

#3(a,b), p.229 Use cup products to show that there is no map $f : \mathbb{R}P^n \to \mathbb{R}P^m$, if $n > m$, inducing a nontrivial map $f^* : H^1(\mathbb{R}P^m, \mathbb{Z}_2) \to H^1(\mathbb{R}P^n, \mathbb{Z}_2) where.$ Here, $H^*(\mathbb{R}P^k;\mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^{k+1})$ where $x \in H^1(\mathbb{R}P^k;\mathbb{Z}_2)$. (If this were to exist, $f^* : H^I(\mathbb{R}P^m;\mathbb{Z}_2) \to H^*(\mathbb{R}P^n;\mathbb{Z}_2)$ is a ring homomorphism...) What is the corresponding results for map $\mathbb{C}P^n \to \mathbb{C}P^m$? (would have to be H_2... (b) says: Prove the Borsuk-Ulam theorem: if $f : \S^n \to \mathbb{R}^n$ then $f(-x) = f(x)$ for some x. Suppose not. Then $f(-x) \neq f(x)$. Define $g : S^n \to \S^{n-1}$,

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}$$

. Then $g(-x) = -g(x)$,so g induces a map $\mathbb{R}P^n \to \mathbb{R}P^{n-1}$. Can we find a lift $\tilde{h}$ in the diagram below?

$$\begin{tikzcd}
S^n \arrow[r, "g"] \arrow[d, "p"'] & S^{n-1} \arrow[d, "p"] \\
\mathbb{R}P^n \arrow[r, "h"] \arrow[ur, "\tilde{h}"] & \mathbb{R}P^{n-1}
\end{tikzcd}$$

#4 p.229 $f : \mathbb{C}P^n \to \mathbb{C}P^n$. Compute $\lambda(f)$ using H^*.

#5 p.229 $H^*(\mathbb{R}P^\infty;\mathbb{Z}_m) \approx Z_m[\alpha,\beta]/(2\alpha, 2\beta, \alpha^2)$ where $\alpha \in H^1, \beta \in H^2$ and $m > 2$. There is a corresponding statement in the book for $m = 2$.

#12 p.229 $X = S^1 \times \mathbb{C}P^\infty / S^1 \times pt$. $Y = S^3 \times \mathbb{C}P^\infty$. Show that $H^*(X;\mathbb{Z}) \approx H^*(Y;\mathbb{Z})$. actually true for any coefficients. One can detect the difference between these using the Steenrod algebra. This problem will require some reading; should use the Kunneth formula.

9.2 Proof of the *UCT* in Cohomology

There exists a short exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \ker h & \to & H^n(X;G) & \xrightarrow{h} & Hom(H_n(X),G)) & \to & 0 \\ & & \| & & & & & & \\ & & Ext(H_{n-1}(X),G) & & & & & & \end{array}$$

Today's order of business: define Ext, and identify it with the kernel.

Definition: (the functor Ext) Let A,G be abelian groups. Choose free abelian groups F_1, F_0 such that there exists a presentation

$$0 \to F_1 \xrightarrow{\partial} F_0 \xrightarrow{\epsilon} A \to 0$$

This is an exact sequence, called a *free presentation* of A. We apply the functor Hom $(\cdot, G)$ to this presentation and then define $Ext(A,G)$ to be the cokernel of ∂^*: $Ext(A,G) :=$ coker ∂^*.

$$\begin{array}{ccccccc} 0 & \leftarrow & \text{coker } \partial^* & \leftarrow \text{Hom }(F_1,G) \xleftarrow{\partial^*} \text{Hom }(F_0,G) \xleftarrow{\epsilon} \text{Hom }(A,G) \leftarrow 0 \\ & & \|\text{def} & \\ & & Ext(A,G) & \end{array}$$

Remark: This defninition of $Ext(A,G)$ suggests that it does not depend on the choice of the resolution. One needs the following lemma to se this:

Lemma: Consider 2 resolutions and a group homomorphism $\phi : A \to A'$:

$$\begin{array}{ccccccccc} 0 & \to & F_1 & \xrightarrow{\partial} & F_0 & \xrightarrow{\epsilon} & A & \to & 0 \\ & & & & & & \downarrow\phi & & \\ 0 & \to & F_1' & \xrightarrow{\partial'} & F_0' & \xrightarrow{\epsilon'} & A & \to & 0 \end{array}$$

Then there exists a chain map $\alpha : F_* \to F_*'$ and moreover, any two such are chain homotopic.

Proof: : Diagram chasing:

$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[r, "\partial"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "\epsilon"] \arrow[d, "\alpha_0"] & A \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & F_1' \arrow[r, "\partial'"] & F_0' \arrow[r, "\epsilon'"] & A \arrow[r] & 0 \end{tikzcd}$$

We must find α_0, α_1 making this diagram commute. To define α_0, pick a generator $x_0 \in F_0$. Look at $\phi\epsilon(X_0)$. There exits $x_0' \in F_0'$ (not necessarily unique) such that

$$\phi\epsilon(x_0) = \epsilon'(x_0').$$

Define $\alpha(x_0) = x_0'$. Extend linearly to a map $\alpha : F_0 \to F_0'$.

Now choose a generator $x_1 \in F_1$. $\epsilon'\alpha_0\partial(x_1) = 0$ (since $\epsilon'\alpha_0\partial = \phi\epsilon\partial = 0$). Thus $\alpha_0\partial(x_1) \in \ker\epsilon'$, so there exists a unique $x_1' \in F_1'$ such that $\partial'(x_1') = \alpha_0\partial(x_1)$. Define $\alpha_1(x_1) = x_1'$, and extend linearly.

Now, suppose that $\alpha, \beta : F_* \to F_*'$ are chain maps; we would like to say that α, β are chain homotopic.

$$\begin{tikzcd} 0 \arrow[r] & F_1 \arrow[r, "\partial"] \arrow[d, "\alpha_1 - \beta_1"'] & F_0 \arrow[r, "\epsilon"] \arrow[d, "\alpha_0 - \beta_0"] \arrow[dl, "D_0"'] & A \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & F_1' \arrow[r, "\partial'"] & F_0' \arrow[r, "\epsilon'"] & A \arrow[r] & 0 \end{tikzcd}$$

Diagram chasing (exercise): There exists $D_0 : F_0 \to F_1'$ such that

$$\partial' D_0 = \alpha_0 - \beta_0, \qquad D_0 \partial = \alpha_1 - \beta_1.$$

D_0 is the chain homotopy from α to β. This implies that $\alpha* = \beta*$ on homology. $\square$

Exercise: $Ext(A, G)$ is well defined.

Universal Coefficient Theorem: There exists a short exact sequece

$$0 \longrightarrow Ext(H_{n-1}(X), G) \longrightarrow H^n(X; G) \xrightarrow{h} Hom(H_n(X), G)) \longrightarrow 0$$

Moreover, this exact sequence is functorial in X and G, and is split (but the splitting is not functorial in X).

Thus $H^n(X; G) \approx \text{Hom}\ (H_n(X), G) \oplus Ext(H_n(X), G)$. This tells us how to compute cohomology.

9.3 Properties of $Ext(A, G)$

- If A is free abelian then $Ext(A, G) = 0$. to see this choose a resolution

$$0 \to F_1 \xrightarrow{\partial} F_0 \xrightarrow{\epsilon} A \to 0$$

  such that $F_0 = A$, $\epsilon = id_A, F_1 = 0$.

- $Ext(A_1 \oplus A_2, G) = Ext(A_1, G) \oplus Ext(A_1, G)$ (splice 2 resolutions).

- $Ext(\mathbb{Z}_n, G) \approx G/nG$. Consider the free resolution

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \xrightarrow{\times n} & \mathbb{Z} & \to & \mathbb{Z}_n & \to & 0 \\ & & || & & || & & & & \\ & & F_1 & & F_1 & & & & \end{array}$$

  Therefore, $Ext(\mathbb{Z}_n, G)$ is determined from the diagram

$$\begin{array}{ccccccc} 0 \leftarrow & Ext(\mathbb{Z}_n, G) & \leftarrow & \text{Hom}\ (Z, G) & \stackrel{\times n}{\leftarrow} & \text{Hom}\ (Z, G) \\ & \downarrow\approx & & \downarrow\approx & & \downarrow\approx \\ & G/nG & \leftarrow & G & \stackrel{\times n}{\leftarrow} & G \end{array}$$

- If $A = Z^n \oplus T$, where T is a finite group then $Ext(A, G) \approx Ext(T, G)$. This follows directly from the properties above.

  Therefore, $Ext(A, G) \approx G/n_1G \oplus \cdots \oplus G/n_kG$.

Example: By the Universal Coefficient Theorem we get:

$$\begin{aligned} H^n(X; Z) &\approx Hom(H_n(X), \mathbb{Z}) \oplus Ext_(H_{n-1}(X), \mathbb{Z}) \\ &\approx \text{free part of } H_n(X) \oplus \text{torsion part of } H_{n-1}(X) \end{aligned}$$

Chapter 10

February 3

10.1 Naturality in the UCT

The Universal Coefficient Theorem is natural for maps $f : X \to Y$, that is the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & Ext(H_{i-1}(X), G) \arrow[r] & H^i(X;G) \arrow[r, "h"] & \text{Hom }(H_i(X), G) \arrow[r] & 0 \\
0 \arrow[r] & Ext(H_{i-1}(Y), G) \arrow[r] \arrow[u, "Ext(f_*, id)"'] & H^i(Y;G) \arrow[r, "h"] \arrow[u, "f^*"'] & \text{Hom }(H_i(Y), G) \arrow[r] \arrow[u, "\text{Hom }(f_*, id)"'] & 0
\end{tikzcd}$$

There also exist splittings

$$s_X : \text{Hom }(H_i(X), G) \to H^i(X;G) \qquad s_X : \text{Hom }(H_i(X), G) \to H^i(X;G),$$

but we can't choose them to be natural with respect to $f : X \to Y$.

Example: #11, p.205. Let X be the *Moore space* $M(\mathbb{Z}_m, n)$, i.e. $X = S^n \cup_f e^{n+1}$, where $f : S^n \to S^n$ is a map of degree m. We will assume $m \neq 0$. Then the cellular chain complex of X is:

$$\begin{array}{ccccccc}
0 \to & C_{n+1} & \longrightarrow & C_n & \to 0 \to \ldots \to 0 \to & C_0 & \to 0 \\
& \| & & \| & & \| & \\
0 \to & \mathbb{Z} & \xrightarrow{\times m} & \mathbb{Z} & \to 0 \to \ldots \to 0 \to & \mathbb{Z} & \to 0
\end{array}$$

So

$$\tilde{H}_i(X) = \begin{cases} \mathbb{Z}_m & i = n \\ 0 & \text{otherwise} \end{cases}$$

In cohomology we have the cochain complex

$$\begin{array}{ccccccc}
0 \leftarrow & C^{n+1} & \longleftarrow & C^n & \leftarrow 0 \leftarrow \ldots \leftarrow 0 \leftarrow & C^0 & \leftarrow 0 \\
& \| & & \| & & \| & \\
& \mathbb{Z} & \xleftarrow{\times m} & \mathbb{Z} & & \mathbb{Z} &
\end{array}$$

and therefore

$$\tilde{H}^i(X) = \begin{cases} \mathbb{Z} & i = n+1 \\ 0 & \text{otherwise} \end{cases}$$

Let $c : X \to S^{n+1}$ be the map that collapses the n-skeleton. Then $c_* : \tilde{H}_i(X) \to \tilde{H}_i(S^{n+1})$ is always 0, but $c^* : \tilde{H}^i(S^{n+1}) \to \tilde{H}^i(X)$ is not 0.

Put $X = M(\mathbb{Z}_m, n)$ in the diagram, $Y = S^{n+1}$, $f = c$, $i = n+1$.

10.2 Proof of the UCT

Consider the short exact sequence of chain complexes:

$$0 \longrightarrow \mathbb{Z}_* \longrightarrow \mathbb{C}_* \longrightarrow \bar{B}_* \longrightarrow 0$$

where in degree n this is

$$0 \to \mathbb{Z}_n \xrightarrow{i} \mathbb{C}_n \xrightarrow{\partial} \bar{B}_{n-1} \to 0$$

- The short exact sequence of chain complexes is split, because all groups are free abelian.
- The boundary operators in Z_* and $\bar{B}_*$ are 0.

Now take Hom $(\cdot, G)$:

$$0 \to \text{Hom }(Z_*, G) \to \text{Hom }(C_*, G) \to \text{Hom }(\bar{B}_*, G) \to 0$$

This is a short exact sequence of cochain complexes, so we get the long exact sequence of cochain complexes:

$$\begin{aligned} \cdots \to \text{Hom }(Z_{n-1}, G) &\xrightarrow{\delta} \text{Hom }(B_{n-1}, G) \to H^n(C; G) \to \\ \text{Hom }(Z_n, G) &\xrightarrow{\delta} \text{Hom }(B_n, G) \to \cdots \end{aligned}$$

$$\begin{aligned} \text{image}\,(H^n(C;G) \to \text{Hom }(Z_n, G)) &= \{\alpha : Z_n \to G \ : \ \alpha|B_n = 0\} \\ &= \text{Hom }(H_n(C), G). \end{aligned}$$

Now let us find Coker Hom $(Z_{n-1}, G) \to \text{Hom }(B_{n-1}, G)$. Consider the exact sequence

$$0 \to B_{n-1} \to Z_{n-1} \to H_{n-1}(C) \to 0$$

Apply Hom $(\cdot, G)$:

$$0 \leftarrow \underset{\underset{\text{coker}}{\parallel}}{Ext(H_{n-1}(C), G)} \leftarrow \text{Hom }(B_{n-1}, G) \leftarrow \text{Hom }(Z_{n-1}, G) \leftarrow \cdots$$

Exercises: Check the details.

10.3 Some Homological Algebra

Let R be a ring with 1, not necessarily commutative.

Definition: A *left R–module* is an abelian group M together with a scalar multiplication $R \times M \to M$, $(r, m) \mapsto rm$, satisfying the following axioms $\forall\, r_1, r_2 \in R, m_1, m_2 \in M$:

(1) $(r_1 + r_2)m = r_1 m + r_2 m$ and $r(m_1 + m_2) = rm_1 + rm_2$.

(2) $(r_1 r_2)m = r_1(r_2 m))$ and $1m = m$.

There is a similar definition for right R-modules.

Definition: Suppose A, B are left R modules. Then an R module homomorphism $\phi : A \to B$ is an abelian group homomorphism commuting with scalar multiplication: $\phi(ra) = r\phi(a)$.

Examples:

(1) If M is a left R-module, then it becomes a right R-module by the defnition $mr = r^{-1}m$. There is no difference between left and right modules if R is commutative.

(2) If $R = \mathbb{Z}$ then a left R-module is merely an abelian group.

(3) If $R = F$ is a field then modules are vector spaces over F.

(4) The integral group ring $\mathbb{Z}[G]$ of a group G is defined to be the set of all finite integral linear combinations of elements of G:

$$\mathbb{Z}[G] := \{n_1 g_1 + n_2 g_2 + \cdots + n_r g_r \mid n_i \in \mathbb{Z}, g_i \in G\}$$

$\mathbb{Z}[G]$ is a ring with 1 with respect to the obvious definitions of addition and multiplication. In a similar way we can define other group rings $R[G]$ or group algebras $k[G]$, where k is a field.

(5) The ring of integers $\mathbb{Z}$ is a trivial $\mathbb{Z}[G]$ module for any group G: define $g \times n = n$ for all $g \in G, n \in \mathbb{Z}$ and extend linearly. There is an augmentation homomorphism $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ defined by $\epsilon(n_1 g_1 + n_2 g_2 + \cdots + n_r g_r) = n_1 + n_2 + \cdots + n_r$. The kernel of this homomorphism is a $\mathbb{Z}[G]$ module, the augmentation ideal $I[G]$.

(6) Given left R-modules A, B consider the abelian group Hom $_R(A, B)$ of R module homomorphisms $\phi : A \to B$. Exercise: Is $Hom_R(A, B)$ an R-module? (it turns out not to be)

(7) Let A be a left R-module, and B be a right R module. Then $B \otimes_R A$ is the free abelian group on pairs (b, a) ($\forall b \in B, a \in A$), modulo the relations:

- $(b_1 + b_2, a) = (b_1, a) + (b_2, a)$
- $(b, a_1 + a_2) = (b, a_1) + (b, a_2)$ (so far, this is the tensor product over the integers)
- $(br, a) = (b, ra)$.

Notation: $b \otimes a$ is the class of (b, a) in $B \otimes_R A$.

Remark: $B \otimes_R A = B \otimes_{\mathbb{Z}} A/(\text{relations } br \otimes_{\mathbb{Z}} a = b \otimes_{\mathbb{Z}} ra)$.

10.4 Group Homology and Cohomology

Let A be a left R-module and B be a right R module, where $R = \mathbb{Z}[G]$. We will define the cohomology of G with coefficients in A and the homology of G with coefficients in B, denoted by $H^*(G; A)$ and $H_*(G; B)$ respectively.

Definition: A free R-resolution of $\mathbb{Z}$ is a sequence free R-modules F_i $(i \geq 0)$ and an exact sequence of R module homomorphisms:

$$\cdots \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\eta} \mathbb{Z} \to 0.$$

Remark: The kernel of η may not be a free R-module, but we can choose some free R module F_1 and an R module homomorphism $\partial_1 : F_1 \to F_0$ whose image is $\ker \eta$. We then repeat this procedure for $ker\partial_1$, that is we choose some free R module F_2 and an R module homomorphism $\partial_2 : F_2 \to F_1$ whose image is $\ker \partial_1$. We repeat this construction over and over. The resolution may well be infinite.

The homology of the chain complex F_* is not interesting:

$$H_i(F_*) = \begin{cases} 0 & \text{if } i > 0 \\ ker(\eta) & \text{if } i = 0 \end{cases}$$

To get something interesting consider the chain complex $B \otimes_R F_*$ and the cochain complex $\text{Hom}_R(F_*, A)$.

Definition: $H_*(G; B) := H_*(B \otimes_R F_*)$ and $H^*(G; A) := H^*(\text{Hom}_R(F_*, A))$.

Exercise: Show that this definition does not depend on the choice of free resolution F_*. The proof will involve constructing chain maps and homotopies from one resolution into another.

Example. Suppose that $p : \tilde{X} \to X$ is a regular covering such that $\tilde{X}$ is contractible and $\pi_1(X) = G$. X is called a $K(G, 1)$ *space*. It is unique up to homotopy. We will relate $C_*(\tilde{X})$ to a free $\mathbb{Z}[G]$ resolution of $\mathbb{Z}$. For argument's sake let's assume X is a simplicial complex. Let $C_*(X)$ denote the simplicial chain complex of X.

Let σ be an n-simplex (i.e. a generator of $C_n(X)$). We consider σ as a mapping $\sigma : \Delta^n \to X$, where Δ^n is the standard n-simplex. Then there exists a lift $\tilde{\sigma} : \Delta^n \to \tilde{X}$ making the following diagram commute:

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \Delta^n \arrow[ur, "\tilde{\sigma}"] \arrow[r, "\sigma"'] & X \end{tikzcd}$$

Now G acts on $\tilde{X}$ by covering transformations and the set of all lifts of σ is exactly $\{g \circ \tilde{\sigma} | g \in G\}$.

Let $\sigma_1, \ldots, \sigma_k$ be the n-simplices in X. Make a fixed choice of a lift $\tilde{\sigma}_i$ for each σ_i. Then a typical element of $C_n\tilde{X}$ is an integral linear combination of the generators $g\tilde{\sigma}_i$, $1 \leq i \leq r, g \in G$. Therefore $C_n(\tilde{X})$ is a free $Z[G]$-module of rank k on the generators $\tilde{\sigma_1}, \ldots, \tilde{\sigma_k}$.

Since $\tilde{X}$ is contractible we have a free $\mathbb{Z}[G]$-resolution of $\mathbb{Z}$:

$$\cdots \to C_3(\tilde{X}) \xrightarrow{\partial_3} C_2(\tilde{X}) \xrightarrow{\partial_2} C_1(\tilde{X}) \xrightarrow{\partial_1} C_0(\tilde{X}) \xrightarrow{\eta} \mathbb{Z} \to 0$$

Therefore the homology and cohomology of the group G are just the homology and cohomology of the space $\tilde{X}$ with the appropriate coefficents: $H_*(G; B) = H_*(\tilde{X}; B)$ and $H^*(G; A) = H^*(\text{Hom } (C_*(\tilde{X}), A))$. In particular, taking A and B to both be the trivial $\mathbb{Z}[G]$ module $\mathbb{Z}$ we see that

$$H_*(G; \mathbb{Z}) = H_*(\tilde{X}; \mathbb{Z}) = H_*(X),\ H^*(G; \mathbb{Z}) = H^*(\text{Hom } (C_*(\tilde{X}), \mathbb{Z})) = H^*(X).$$

Exercise: Show that the chain complex $\mathbb{Z} \otimes_{\mathbb{Z}[G]} C_*(\tilde{X}) \approx C_*(X)$ and the cochain complex Hom $_{\mathbb{Z}[G]}(C^*(\tilde{X}), \mathbb{Z}) \approx \text{Hom } (C^*(X), \mathbb{Z})$.

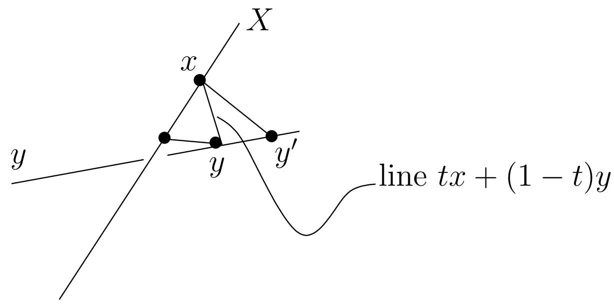
10.5 The Milnor Construction

In this section we show how to construct a particular model of a space of type $K(G, 1)$, where G is a discrete group. Recall the definition of the join of 2 spaces:

Definition: The *join* of two spaces X, Y is the quotient space

$$X * Y * I \Bigg/ \left\{ \begin{array}{c} (x, y, 0) = (x', y, 0) \\ (x, y', 1) = (x, y', 1) \\ \forall x, x' \in x, y, y' \in Y \end{array} \right\}$$

Imagine $X, Y \subset \mathbb{R}^N$, $N >> 0$, so that any 2 distinct line segments $tx + (1 - t)y, tx' + (1 - t)y', 0 \leq t \leq 1, x, x' \in X, y, y' \in Y$ meet only at a common endpoint. Then the picture is



We can iterate this construction, taking the join of 3 spaces: $X * Y * Z := (X * Y) * Z$. This operation is associative $(X * Y) * Z \approx X * (Y * Z)$. We can also think of the 3-fold join $X * Y * Z$ as the space of all 2 simplexes joining arbitrary points $x \in X, y \in Y, z \in Z$, with the only intersections coming from common boundaries. Then we can represent a point in $X * Y * Z$ as a convex linear combination $t_1x + t_2y + t_3z$. It is then clear how to define interated joins $X_1 * X_2 * \cdots * X_n$.

Definition: If G is a group then $E_n(G) := G * G * \cdots * G$ (n copies). A typical point in $E_n(G)$ is written as

$$t_1g_1 + t_2g_2 + \cdots + t_ng_n, \text{ where } t_i \geq 0, g_i \in G, 0 \leq i \leq n \text{ and } t_1 + \cdots t_n = 1.$$

The group G acts on itself by left multiplication and therefore there exists a diagonal action of G on $E_n(G)$ obtained by linear extension to n simplices:

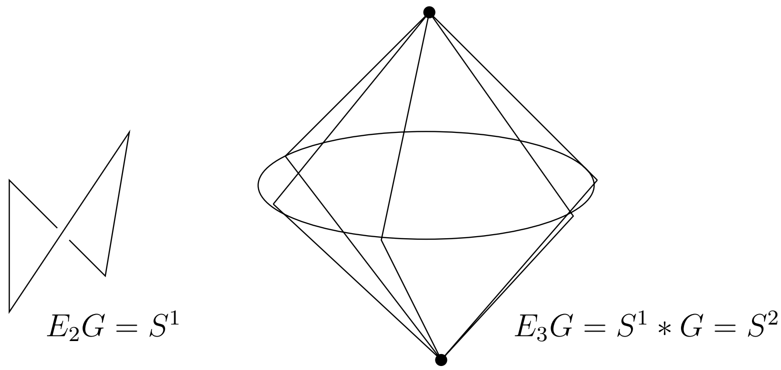
$$g \times (t_1g_1 + t_2g_2 + \cdots + t_ng_n) = t_1gg_1 + t_2gg_2 + \cdots + c_ngg_n.$$

Definition: The quotient space by the action of G on $E_n(G)$ is denoted $B_n(G) := E_n(G)/G$.

There are natural inclusions $E_n(G) \subset E_{n+1}(G)$, compatible with the G action, and therefore there are also natural inclusions inclusions $B_n(G) \subset B_{n+1}(G)$.

Definition: $E(G) := \bigcup_{n\geq1} E_n(G)$ and $BG := \bigcup_{n\geq1} B_n(G) = EG/G$.

Example: $G = \mathbb{Z}_2$.



More generally, $E_nG = S^{n-1}$ for all n. Moreover, $E_G = S^\infty$ is contractible and $BG = \mathbb{R}P^\infty$.

Theorem: EG is contractible.

Chapter 11

February 8

11.1 A Seminar by Joseph Maher

Questions:

1. Which finite groups G admit a fixed-point free action on S^3? This is still an openproblem.

2. Which finite groups G admit a fixed-point free linear action on S^3? In other words which finite groups admit representations $\rho : G \to O(4)$ such that $\rho(g)$ does not have $+1$ as an eigenvalue for $g \in G, g \neq e$?

   Recall that $O(4)$ is the group of 4×4 orthogonal matrices. It is the group of rigid motions of the 3 sphere. Eigenvalue $+1$ means that $\rho(g) \cdot v = v$ for some eigenvector v. This question has been answered, 60 years ago.

3. Suppose that G admits a fixed-point free linear action on S^3 (e.g. $G \approx \mathbb{Z}_n$). Suppose there exists some topological action of G on S^3, without fixed points. Is the topological action conjugate to a linear one?

$$\begin{array}{ccc} S^3 & \xrightarrow{\text{linear}} & S^3 \\ h \uparrow \approx & & h^{-1} \downarrow \approx \\ S^3 & \xrightarrow{\text{top. action}} & S^3 \end{array}$$

Livesay: circa 1950. Yes, for $G = \mathbb{Z}_2$.

Maher: Yes for $G = \mathbb{Z}_3$.

Perelman: Announced a proof of the Thurston Geometrization Conjecture. If the proof is correct it would imply the Poincaré conjecture. It would also follow that if a group acted fixed point freely on S^3 then it would act fixed point freely and linearly, and moreover, the topological action would be conjugate to a linear one.

11.2 Products

Let R be a ring with 1. There are three types of products.

1. The cup product: $H^i(X; R) \times H^j(X; R) \to H^{i+j}(X; R)$, $(u, v) \mapsto u \cup v$.
2. The cross product: $H^i(X; R) \times H^j(Y; R) \to H^{i+j}(X \times Y; R)$, $(u, v) \to u \times v$.
3. The cap product: $H_i(X; R) \times H^j(X; R) \to H_{i-j}(X; R)$, $(u, v) \mapsto u \cap v$.

All 3 products are bilinear in u, v and so induce products on the appropriate tensor products.

Motivation: Suppose we try to find a "cup product" in homology:

$$H_i(X; R) \times H_j(X; R) \to H_{i+j}(X; R).$$

Indeed, there exists a cross product in homology, $H_i(X;R) \times H_j(y;R) \to H_{i+j}(X \times Y;R)$. Let e^i_α be an i-cell in X and e^j_β b e a j-cell in Y. Then $e^i_\alpha \times e^j_\beta$ is an $i+j$ cell in $X \times Y$. The theory says that this passes to a cross product in homology.

Now specialize to $X = Y$: $H_i(X;R) \times H_j(X;R) \to H^{i+j}(X \times X;R) \xrightarrow{?} H_{i+j}(X;R)$. The first map is the cross product in homology, but there's no really good map from $X \times X$ to X which would yield a reasonable notion of a product.

In cohomology the situation is different:

$$H^i(X;R) \times H^j(X;R) \xrightarrow{\times} H^{i+j}(X \times X;R) \xrightarrow{\Delta^*} H^{i+j}(X;R),$$

where $\Delta : X \to X \times X$ is the diagonal. this yields a reasonable definition of a product. However, we will adopt a different definition.

Definition of the cup product: $H^i(X;R) \otimes H^j(X;R) \to H^{i+j}(X;R)$. Take cohomology classes $u \in H^i(X;R), v \in H^j(X;R)$ in singular cohomology and represent them by cocycle maps: $\alpha : C_i(X) \to R$ and $\beta : C_j(X) \to R$ respectively.

Then $\alpha \cup \beta$ will be the homomorphism $\alpha \cup \beta : C_{i+j}(X) \to \mathbb{R}$ defined as follows: choose a generator $\sigma : \Delta^{i+j} \to X$ for the singular chain group $C_{i+j}(X)$, where Δ^{i+j} is a standard $i+j$ simplex with vertices $v_0, v_1, \ldots, v_{i+j}$. Then $\alpha \cup \beta(\sigma) := \alpha(\sigma|\Delta^i) \cdot \beta(\sigma|\Delta^j)$, where Δ^i and Δ^j are the standard simplexes spanned by $[v_0, \ldots, v_i]$ and $[v_i, \ldots, v_{i+j}]$ respectively. The product $\alpha(\sigma|_{\Delta^i}) \cdot \beta(\sigma|_{\Delta^j})$ is just the ring product in R. Finally we extend this linearly to a homomorphism $\alpha \cup \beta : C_{i+j}(X) \to \mathbb{R}$.

Exercise: Let δ denote any of the coboundary operators $\delta : C^n \to C^{n-1}$. Then

$$\delta(\alpha \cup \beta) = \delta(\alpha) \cup \beta + (-1)^i \alpha \cup \delta(\beta).$$

From this it follows easily that the cup product of cochains yields a cup product in cohomology,

$$H^i(X;R) \otimes H^j(X;R) \to H^{i+j}(X;R),\ (u, v) \mapsto u \cup v.$$

Definition: The total cohomology of X with coefficients in R is $H^*(X;R) := \oplus_{i\geq 0} H^i(X;R)$.

Properties of the cup product:

1. $H^*(X;R)$ becomes a ring with respect to the cup product. In particular $u \cup v$ is bilinear in u and v. Moreover $\exists$ an identity $1 \in H^0(X;R)$, namely the class that takes the value id_R on every point $x \in X$. Recall that $C_0(X)$ is the free abelian group on the points of X, and $C^0(X) = \text{Hom}\,(C_0(X), R)$.
2. If $f : Y \to X$, then $f^* : H*(Y;R) \to H*(X;R)$ is a ring homomorphism: that is $f^*(u \cup v) = f^*(u) \cup f^*(v)$, where $u \in H^i(Y;R)$ and $v \in H^j(Y;R)$.
3. The cup product is graded commutative: $u \cup v = (-1)^{ij} v \cup u$ if $u \in H^i(X;R), v \in H^j(X;R)$.
4. $\exists$ a relative form of the cup product: $H^i(X,A;R) \otimes H^j(X,B;R) \to H^{i+j}(X, A\cup B;R)$.

Example (The Hopf invariant one problem): Suppose $f : S^{2n-1} \to S^n$ is some map and $X = S^n \cup_f e^{2n}$, $n > 1$. Then

$$H^i(X;\mathbb{Z}) = \begin{cases} Z & i = 0, n, 2n \\ 0 & \text{otherwise} \end{cases}$$

Let $u \in H^n(X;\mathbb{Z}), v \in H^{2n}(X;Z)$ be generators. Then $u \cup u \in H^{2n}(X;\mathbb{Z})$ and therefore, $u \cup u = kv$, for some integer k, called the *Hopf invariant* $H(f)$ *of* f. $H(f)$ is unique up to sign; it depends on v but not u.

Theorem: $H(f) = 0$ if n is odd.

Proof: By graded commutativity of the cup product we have $u \cup u = (-1)^{n \cdot n} u \cup u = -u \cup u$, so $u \cup u = 0$ ($\mathbb{Z}$ has no 2-torsion).

Remark: Hopf proved that $\exists$ elements of Hopf invariant 1 for $n = 2, 4, 8$.

Next time we will use the mod 2 Steenrod algebra to show that if $f : S^{2n-1} \to S^n$ has Hopf Invariant one then n must be a power of 2.

Theorem: If $f : S^{2n-1} \to S^n$ has odd Hopf invariant, then n is a power of 2.

Definition of the cross product: $H^i(X;R) \otimes H^j(Y;R) \xrightarrow{\times} H^{i+j}(X \times Y;R),\ u \otimes v \to u \times v.$

Let $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ be the projections onto X and Y respectively. Then the cross product is defined by

$$u \times v := p_1^*(u) \cup p_2^*(v), \text{ where } u \in H^i(X;R), v \in H^j(Y;R).$$

Properties of the cross product then follow from corresponding properties of the cup product.

Definition of the cap product: $H_i(X;R) \otimes H^j(X;R) \xrightarrow{\cap} H_{i-j}(X;R).$

Let σ be a generator of the singular chain group $C_i(X)$. Thus σ is a continuous mapping $\sigma : \Delta^i \to X$, where Δ^i is a standard i simplex. Let the vertices of Δ be $v_0, \ldots, v_i$. If α is a singular cochain $\alpha : C_j(X) \to R$ then $\sigma \cap \alpha$ is the chain in $C_{i-j}(X)$ defined by

$$\sigma \cap \alpha = \alpha(\sigma|_{\Delta^j}) \cdot \sigma|_{\Delta^{i-j}},$$

where Δ^i is the standard i simplex spanned by the vertices $v_0, \ldots, v_i$ and Δ^{i-j} is the standard $i-j$ simplex spanned by the vertices $v_j, \ldots, v_i$.

Remarks: $\alpha(\sigma|_{\Delta^j}) \in R$ and $\sigma|_{\Delta^{i-j}} \in C_{i-j}(X)$. Thus $\sigma \cap \alpha \in C_{i-j}(X;R)$.

From the definition it follows that the cap product is bilinear. The next Lemma shows that the cap product defined on the chain/cochain level passes to a cap product on the homology/cohomology level:

$$H_i(X;R) \otimes H^j(X;R) \xrightarrow{\cap} H_{i-j}(X;R),\ u \otimes v \to u \cap v.$$

.

Lemma: $\partial(\sigma \cap \alpha) = (-1)^j(\partial(\sigma) \cap \alpha - \sigma \cap \delta(\alpha))$, where ∂ and δ are boundary or coboundary operators.

Chapter 12

February 23

12.1 Assignment 2

Assignment 2 is on the web. There is a typo in #5, p.229 of this week's homework, and thus the due date is extended to next Tuesday. Change to

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}_{2k}) \approx Z_{2k}[\alpha, \beta]/(2\alpha, 2\beta, \alpha^2 - k\beta)$$

where $\alpha \in H^1(\mathbb{R}P^\infty; Z_{2k}), \beta \in H^2(\mathbb{R}P^\infty; \mathbb{Z}_{2k})$.

#12, p.229: $X = S^1 \times \mathbb{C}P^\infty / X^1 \times pt$, $Y = S^3 \times \mathbb{C}P^\infty$.

One needs to use the relative cross product. See Theorem 3.21. There exist isomorphisms

$$H^*(X^1) \otimes H^*(\mathbb{C}P^\infty, pt) \xrightarrow{\times} H^*(S^1 \times \mathbb{C}P^\infty, S^1 \times pt) \approx \tilde{H}^*(X)$$

$$H^*(S^3) \otimes H^*(\mathbb{C}P^\infty) \xrightarrow{\times} H^*(S^3 \times \mathbb{C}P^\infty) \approx H^*(Y)$$

Let $1 \in H^0(S^1)$ denote the identity, $x \in H^1(S^1) \approx \mathbb{Z}$ be a generator and $y \in H^2(\mathbb{C}P^\infty, pt) \approx \mathbb{Z}$ a generator.

The generators of $H^*(S^1) \otimes H^*(\mathbb{C}P^\infty, pt)$ are $1 \otimes y^k \in H^0(S^1) \otimes H^{2k}(\mathbb{C}P^\infty, pt)$ for $k \geq 1$ and $x \otimes y^k \in H^1(S^1) \otimes H^{2k}(\mathbb{C}P^\infty, pt)$ for $k \geq 1$.

Corresponding to these classes we have

$$1 \times y^k \in H^{2k}(X) \approx \mathbb{Z}, k \geq 1, \text{ and } x \times y^k \in H^{2k+1}(X) \approx \mathbb{Z}, k \geq 1.$$

Therefore

$$\tilde{H}^i(X) = \begin{cases} Z & i = 2, 3, 4, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Next we determine the generators of $H^*(S^3) \otimes H^*(\mathbb{C}P^\infty)$. Let $1 \in H^0(S^3)$ denote the identity, $u \in H^3(S^3)$ be a generator and $v \in H^2(\mathbb{C}P^\infty)$ a generator.

So we get a generator $1 \otimes v^{2k} \in H^0(S^3) \otimes H^{2k}(\mathbb{C}P^\infty) \approx \mathbb{Z}$ and a generator $u \otimes v^{2k} \in H^3(S^3) \otimes H^{2k}(\mathbb{C}P^\infty) \approx \mathbb{Z}$. Then $1 \times v^{2k} \in H^{2k}(S^3 \times \mathbb{C}P^\infty)$ is a generator $\approx Z$, and $u \times v^{2k} \in H^{3+2k}(S^3 \times \mathbb{C}P^\infty) \approx \mathbb{Z}$ is a generator.

We must show that $H^*(Y) \approx H^*(X)$ as rings. We know that

$$H^i(X) \approx H^i(Y) \approx \begin{cases} Z & i = 0, 2, 3, 4, \dots \\ 0 & \text{otherwise} \end{cases}$$

Moreover, we know what the generators are. Therefore, we define a group homomorphism $\theta : H^*(X) \to H^*(Y)$ by mapping generators to generators. Now, check that there is a ring map.

12.2 Examples of computing cohomology rings

#1 p.228 Consider a genus g surface M_g. Map it into a one-point union of tori $T \vee T \vee \cdots \vee T$, by collapsing part of M_g to a point via a map c. Use this to compute $H^*(M_g)$. We know what the homology group structure is, via a CW complex decomposition:

$$H_i(M_g) = \begin{cases} Z & i = 0, 2 \\ Z^{2g} & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Because all of these are free, we can read off the cohomology structure using the UCT.

$$H^i(M_g) = \begin{cases} Z & i = 0, 2 \\ Z^{2g} & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Now pick generators for homology: $a_1, b_1, \ldots, a_g, b_g$.

Now let us compute $H^*(T)$. There are generators $a, b \in H_1(T) = \mathbb{Z} \otimes \mathbb{Z}$ and dual gereators $\alpha, \beta \in H^1(T) \approx \mathbb{Z} \otimes \mathbb{Z}$: $\alpha(a) = 1, \alpha(b) = 0, \beta(a) = 0, \beta(b) = 1$.

Let X_α be a space for each $\alpha \in A$ and let $i_\alpha : X_\alpha \to \bigsqcup_{\alpha \in A} X_\alpha$ denote the inclusion, where A is some index set. Then there is an isomorphism

$$H^*(\bigsqcup_{\alpha \in A} X_\alpha; R) \xrightarrow{\approx} \prod_{\alpha \in A} H^*(X_\alpha; R), \; u \to (i_\alpha^*(u))_{\alpha \in A}$$

The left hand side is a ring with respect to cup product. Moreover the map $i_\alpha^* : H^*(\coprod_{\alpha\in A} X_a; R) \to H^*(X_\alpha; R)$ is a ring map for all $\alpha \in A$.

Therefore, $H^*(\coprod_{\alpha\in A} X_\alpha; R) \xrightarrow{\approx} \prod_{\alpha\in A} H^*(X_\alpha; R)$ is a ring isomorphism. The ring structure on the right hand side is given by coordinate-wise multiplication.

In reduced cohomology the inclusions i_α induce a ring isomorphism

$$\tilde{H}^*(\bigvee_{\alpha\in A} X_\alpha; R) \xrightarrow{\approx} \prod_{\alpha\in A} \tilde{H}^*(X_\alpha; R)$$

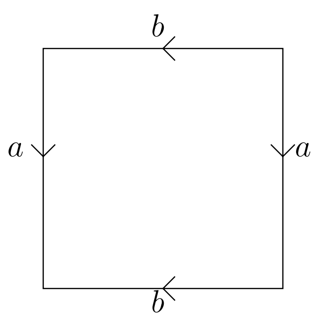
If $|A| = 2$ then $\tilde{H}^*(X \vee Y; R) \approx \tilde{H}^*(X; R) \oplus \tilde{H}^*(Y; R)$ are isomorphic as rings.

$u \in H^i(X), v \in H^j(Y)$. We have projections $X \vee Y \xrightarrow{p} X$, $X \vee Y \xrightarrow{p} Y$, where $p^*(u) \in \tilde{H}^i(X \vee Y), q^*(v) \in \tilde{H}^j(X \vee Y)$. Here, $p^*(u) \to (u, 0-)$ and $q^*(v) \to (0, v)$, so therefore, $p^*(u) \cup p^*(v) \to (u, 0) \cdot (0, v) = (0, 0)$.

In particular, this means that $\alpha_i \cup \alpha_j = \alpha_i \cup \beta_j = 0$ when $i \neq j$.

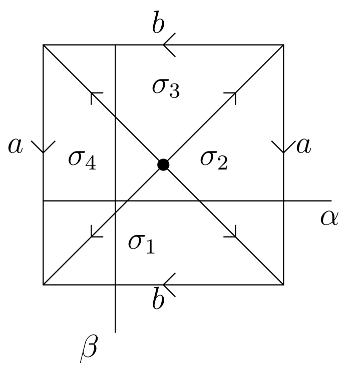
Also, $\alpha_i \cup \alpha_i = 0$ in the torus, so it is true here as well. Also $\alpha_i \cup \beta_i = \pm 1$, for the same reason.

Example: $H^*(T; Z)$. a and b are generators for $H_1(T) \approx \mathbb{Z} \otimes \mathbb{Z}$.



α, β are the dual classes in $H^1(T) \approx \text{Hom } (H_1(T), \mathbb{Z})$.

Now, let us make a Δ-complex structure:



α is represented by the cocycle ϕ taking the value $+1$ on those edges which meet α, and 0 on the other edges. (Exercise: $\delta(\phi) = 0$)

β is represented by the cocycle ψ taking the value $+1$ on those edges meeting α and 0 on all others. Exercise: $\delta(\psi) = 0$.

Therefore, $\phi \cup \psi$ is a cocycle representing $\alpha \cup \beta$. A generator for $H_2(T) \approx \mathbb{Z}$ is the sum $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4$. Exercise: $\phi \cup \psi(\text{this generator}) = \pm 1$.

So

$$\phi \cup \psi(\sigma_i) = \begin{cases} 0 & i = 1, 2, 3 \\ \pm 1 & i = 4 \end{cases}$$

12.3 Cohomology Operations

There is a lovely book called "Cohomology Operations" (Norman Steenrod), annals of Math. Studies. Its moral is that one can study the Steenrod algebra axiomatically.

Definition: Let m, n be fixed integers and let G, H be fixed abelian groups. A *cohomology operation* is a family of mappings $\theta_X : H^m(X; G) \to H^n(X; H)$, natural in X.

Example: θ is the cup square.

$$\begin{aligned} H^m(X;R) &\longrightarrow H^2m(X;R) \\ u &\longmapsto u \cup u \end{aligned}$$

Example: Let $\phi : G \to H$ be a group homomorphism. Then there exists an induced map $\phi_* : H^m(X;G) \to H^m(X;H)$.

12.4 Axioms for the mod 2 Steenrod Algebra

There exists a sequence of cohomology operations

$$\mathrm{Sq}^i : H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2)$$

defined for $i = 0, 1, 2, \ldots$, and all n, satisfying the following axioms (not all independent, but we don't know this yet)

1. Sq^0 is the identity.
2. Sq^i is natural in X (part of the definition of cohomology operation, really)
3. $Sq^i(\alpha+\beta) = Sq^i(\alpha) + Sq^i(\beta)$. (Therefore, Sq^i is a group homomorphism). Note that the cup square is quadratic, not linear, except in mod 2.
4. $$Sq^i(\alpha) = \begin{cases} \alpha \cup \alpha & |\alpha| = i \\ 0 & i > \alpha \end{cases}$$
5. $$Sq^i(\alpha \cup \beta) = \sum_{k=0}^{i} Sq^k(\alpha) \cup Sq^{i-k}(\beta)$$

. This is called the Cartan formula. One way to think of this is to think of the graded ring $H^*(X;\mathbb{Z}_2) = \bigoplus_{i\geq 0} H^i(X;\mathbb{Z}_2)$. Then the Cartan formula says that total Steenrod square Sq is a ring homomorphism.

6. Sq^i is *stable*: that is, it commutes with suspension:

$$\begin{tikzcd} H^n(X;\mathbb{Z}_2) \arrow[r, "Sq^i"] \arrow[d, "\approx"', "\sigma"] & H^{n+i}(X;\mathbb{Z}_2) \arrow[d, "\approx"', "\sigma"] \\ H^n(\Sigma X;\mathbb{Z}_2) \arrow[r, "Sq^i"] & H^{n+1+i}(\Sigma X;\mathbb{Z}_2) \end{tikzcd}$$

7. The Adem Relations (Jose Adem). If $a < 2b$ then

$$Sq^a \cdot Sq^b = \sum_j \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$$

8. Sq^1 is the Bockstein (next time)

Chapter 13

February 25

13.1 Axiomatic development of Steenrod Algebra

- One of the axioms for the Steenrod Algebra was the *Adem Relations:* If $a < 2b$ then
$$Sq^a \circ Sq^b = \sum_j \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j$$
Sq^j raises the degree by j, so this rule addresses the situation where we are composing two squares:
$$H^p(X;\mathbb{Z}_2) \xrightarrow{Sq^b} H^{p+b}(X;\mathbb{Z}_2) \xrightarrow{Sq^a} H^{p+a+b}(X;\mathbb{Z}_2)$$
- We take the binomeal coefficient $\binom{b-j-1}{a-2j}$ mod 2.
- Sq^1 is the Bockstein associated to the shorte exact sequence
$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_2 \arrow[r] & \mathbb{Z}_4 \arrow[r] & \mathbb{Z}_2 \arrow[r] & 0 \\ & & \text{gen} \arrow[r, mapsto] & \text{gen} & \end{tikzcd}$$
The Bockstein asdsociated to a short exact sequecne of abelian groups
$$0 \longrightarrow K \xrightarrow{\alpha} E \xrightarrow{\beta} Q \longrightarrow 0$$

Then there exists a long exaqct sequence in cohomology

$$\cdots \longrightarrow H^i(X;K) \xrightarrow{\alpha^*} H^i(X;E) \xrightarrow{\beta^*} H^i(X;Q) \xrightarrow{\delta} H^{i+1}(X;K) \longrightarrow \cdots$$

The Bockstein is the connecting homomorphism δ. Let us see why this is so.

Claim: There exists a short exact sequence of cochain complexes:

$$0 \longrightarrow C^*(X,K) \xrightarrow{\alpha_*} C^*(X,E) \xrightarrow{\beta_*} C^*(X,Q) \longrightarrow 0$$

$\ker \beta_* = Im\alpha_*$:

$$\begin{tikzcd}
 & & C_n(X) \arrow[dl, "\psi"'] \arrow[d, "\phi"] \arrow[dr, "\beta\circ\phi=0"] & & \\
0 \arrow[r] & K \arrow[r, "\alpha"] & E \arrow[r, "\beta"] & Q \arrow[r] & 0
\end{tikzcd}$$

$C_n(X)$ free abelian, so there exists a $\phi : C_n(X) \to K$ such that $\alpha \circ \psi = \phi$. By general theore, there exists a long exact sequence in cohomology.

Lemma: If n is not a power of 2, then there exists a factorization

$$Sq^n = \sum_{j=1}^{n-1} a_j Sq^{n-j} \circ Sq^j.$$

Proof: Set $n = a + b, a < 2b$. Suppose the 2-adic expansion of n is $n = i_0 + i_1 \cdot 2 + i_2 \cdot 2^2 + \cdots + i_k \cdot 2^k$. Write $b = 2^k, a = n - 2^k$.

We need the coefficient of Sq^n in the right hand side to be nonzero. Write $b - 1 = 1 + 2 + \cdots + 2^{k-1}$. Is $\binom{b-1}{a} \equiv 1 \pmod 2$?

A theorem from number theory. Given k adic exapansions

$$\begin{aligned} k &= k_0 + k_1 \cdot 2 + k_2 \cdot 2^2 + \cdots \\ l &= l_0 + l_1 \cdot 2 + l_2 \cdot 2^2 + \cdots \\ \implies \binom{k}{l} &= \binom{k_0}{l_0}\binom{k_1}{l_1}\cdots \pmod 2 \end{aligned}$$

This is nonzero if and only if every $k_i \geq l_i$. Therefore, with these choices, the Adem relations imply $Sq^n = \sum_{j=1}^{n-1} a_j Sq^{n-j} Sq^j$. $\square$

13.2 Application: Hopf invariant 1 problem

Application. If $f : S^{2n-1} \to S^n$ has $H(f) = 1$, then n is a power of 2.

Proof: Definition of the Hopf invariant of $f : S^{2n-1} \to S^n$: let

$$X = S^n \cup_f e^{2n}$$

. Then

$$H^i(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, n, 2n \\ 0 & \text{otherwise} \end{cases}$$

Then $u \in H^n(X;Z)$ is a generator, $v \in H^{2n}(X;\mathbb{Z})$ is a generator. Then $u^2 = k \cdot v$. Then k is the Hopf invariant $H(f)$.

$\bar{u} \in H^n(X;\mathbb{Z}_2) \approx \mathbb{Z}_2$, $\bar{v} \in H^{2n}(X;\mathbb{Z}_2) \approx \mathbb{Z}_2$ By one of the axioms of the Steenrod algebra, in mod 2 cohomology,

$$\bar{u} \cup \bar{u} = Sq^n(\bar{u}) \overset{\text{lemma}}{=} \sum_{j=1}^{n-1} a_j Sq^{n-j} Sq^j.$$

All of the composites $Sq^{n-j}Sq^j$ factor through zero groups. $\square$.

Of course, we would need to demonstrate that there actually is a Steenrod algebra!

13.3 Poincaré Duality

The statement is: If M^n is a closed, connected and orientable manifold, then of dimension n, then

$$H^i(M;\mathbb{Z}) \xrightarrow{\approx} H_{n-i}(M;Z)$$

for all i.

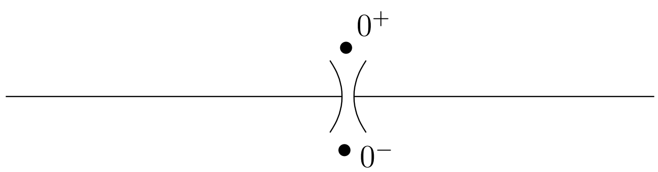
Definition: A manifold of dimension n is a paracompact Hausdorff space M^n , together with an open covering $\mathcal{U} = \{U_i | i \in I\}$ and homeomorphisms $\phi_i : I_i \to \mathbb{R}^n$.

(a) Each U_i is an open, nonempty subset of M.

(b) $\bigcup_{i\in I} U_i = M$.

Definition: A topological space X is *Hausdorff* if, given any two distinct points $x, y \in X$, there exist open sets $U, V \subseteq X$ such that $x \in U, y \in V, U \cap V = \emptyset$.

Example: $X =$ the real line, with the origin doubled.



Topology on X the neighbourhoods of $x \in X$, $x \neq 0^\pm$ are as usual, so that $X - \{0^\pm\}$ is homeomorphic to $\mathbb{R} - \{0\}$.

A neighbourhood of $0^\pm$ is $\{x \in \mathbb{R} | x \neq 0, -\epsilon < x < \epsilon\} \cup \{0^\pm\}$.

This is not Hausdorff.

Example of something which is not paracompact: (the long line) Let Ω be the first uncountable ordinal. Write down all the ordinals before it: $0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots, 2\omega, \ldots, \Omega$, and join each one to the next with line segments. Put the ordinary topology on this space.

A major problem in topology: Classify manifolds in low dimensions.

In dimension 0, manifolds are collections of points with the discrete topology.

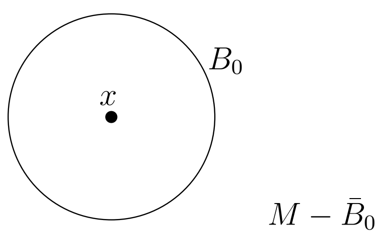
Dimension 1: manifolds are disjoint unions of S^1 and $\mathbb{R}$. If $\partial M \neq 0$ (M is a manifold with boundary) then we also have various types of intervals.

Dimension 2: The orientable closed connected ones are the sphere, the torus, and the genus g surface (g-holed doughnut). The nonorientable ones are $\mathbb{R}P^2$, $\mathbb{K}^2$, and so forth.

13.4 Orientability

M^n will be a closed, connected manifold of dimension n. *Closed* means M is compact and has no boundary.

A *local orientation* on M: take $x \in M$. Then x has a neighbourhood B_0 be an n-ball with $x \in \text{int}(B_0)$.



Here, B_0 is homeomorphic to $\{x \in \mathbb{R}^n \ : \ |x| \leq 1\}$. Now,

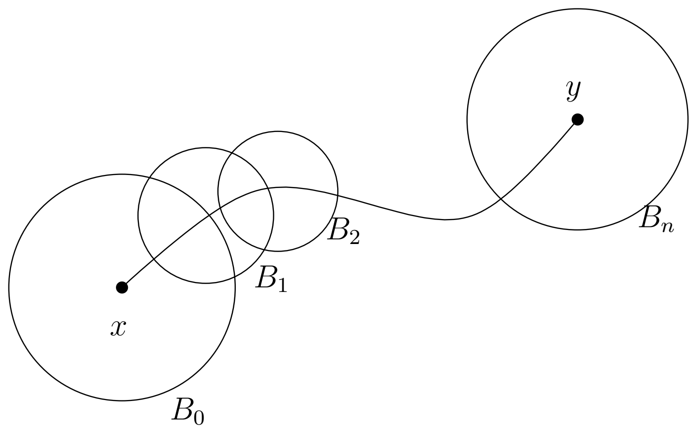
$$\begin{aligned} H_n(M, M - X) &\approx H_n(R^n, R^n - X) \quad \text{Excision – cut out } M - B_0 \\ &\approx H_n - 1(R^n - X) \\ &\approx \mathbb{Z} \end{aligned}$$

Therefore, there exists a generator $\mu_X \in H_n(M, M - \{x\}) \approx \mathbb{Z}$.

Let B be any ball in X (so that $B \approx \{x \in \mathbb{R}^n \ : \ |x| \leq 1\}$). Then $H_n(M, M - B) \approx \mathbb{Z}$ Choose a generator $\mu_B \in H_n(M, M - B)$. Choose a genorator $\mu_B \in H_n(M, M - B)$. If $x, y \in \text{int } B_0$ we can choose the generators μ_x, μ_y, μ_B compatibly, so that

$$\begin{tikzcd} H_n(M, M - \{x\}) & H_n(M, M - B) \arrow[l, "i_*"', "\approx"] \arrow[r, "j_*", "\approx"'] & H_n(M, M - \{y\} \\ \mu_x & \mu_B \arrow[l, mapsto] \arrow[r, mapsto] & \mu_y \end{tikzcd}$$

This is similar to the situation of "analytic continuation":



γ is a path $x \to y$

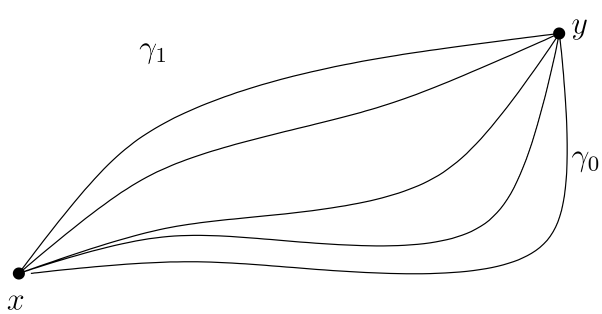
Choose an orientation μ_{B_0} for $H_n(M, M - B_0)$. Then choose an orientation μ_{B_1} for $H_n(M, M - B_1)$ which is compatible with μ_{B_0}. Continue in this way to get a local orientation at y.

Comments:

(a) The choices of the B_i (for a fixed path γ) do not matter.

(b) But the local orientation at y may depend on γ. **Definition:** M^n is *orientable* if all local orentations are compatible.

(c) Suppose γ_0, γ_1 are paths from x to y, and γ_0 is homotopic to γ_1 through paths from x to y.



Then "analytic continuation" of μ_X to μ_Y has the same answer for all paths in the homotopy. This uses compactness, and the proof is the same as the complex analytic one.

Corollary: if M is simply connected, then M is orientable.

Chapter 14

March 1

14.1 Last time

M^n is a closed, connected n-manifold, i.e. compact and without boundary.

Local orientations:

$$\begin{aligned} x \in M : H_n(M, M-x; \mathbb{Z}) &\approx H_n(B, B-x; \mathbb{Z}) \\ &\approx H_n(\mathbb{R}^n, \mathbb{R}^n - x; \mathbb{Z}) \\ &\approx \tilde{H}_{n-1}(R^n - X; \mathbb{Z}) \\ &\approx \mathbb{Z} \end{aligned}$$

Here, B is an n-ball, and $x \in \operatorname{int} B$. Therefore, we get a generator $\mu_x \in H_n(M, M-X; \mathbb{Z})$.

Remarks:

1. We can do this for coefficients in $\mathbb{R}$.
2. If B_0 is an n-ball, then there exists a generator $\mu_{B_0} \in H_n(M, M - B_0; \mathbb{Z}) \approx \mathbb{Z}$

The generators μ_x, μ_y can be chosen compatibly throughout B_0: we have

$$\begin{array}{ccccc} H_n(M, M-X;\mathbb{Z}) & \underset{\approx}{\xleftarrow{i_*}} & H_n(M, M-B_0;\mathbb{Z}) & \underset{\approx}{\xrightarrow{j_*}} & H_n(M, M-y;\mathbb{Z}) \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \\ \mu_X & \longleftarrow & \mu_{B_0} & \longmapsto & \mu_y \end{array}$$

Remark: We can continue a choice of a generator $\mu_x \in H_n(M, M-x)$ along a path γ from x to y. This process of continuation does not depend on either the discs chosen or on the homotopy class of γ.

Definition: M is *orientable* if continuation of local orientation always gives compatible answer at y, for all y in M. ($\Longleftrightarrow$ every loop γ at x gives the same local orientation after continuing)

14.2 Orientable coverings

Definition: Define a (possibly zero) group homomorphism $\theta : \Pi_1(M, x) \to Z_2$ by

$$\theta(\gamma) = \begin{cases} 0 & \text{if continuing along } \gamma \text{ preserves local orientation} \\ 1 & \text{if continuing along } \gamma \text{ reverses local orientation} \end{cases}$$

Note that continuing along γ gives either $\mu_X or - \mu_X$ as a local orientation at x.

Remarks: θ is the zero homomorphism if M is orientable.

Assume that M is not orientable. Then $\theta(\Pi_1(M, x)) = \mathbb{Z}_2$. There exists a kernel:

$$1 \longrightarrow G \longrightarrow \pi_1(M, X) \xrightarrow{\theta} \mathbb{Z}_2 \longrightarrow 1$$

Then, from covering space theory, there exists a covering $p : \tilde{M} \xrightarrow{2:1} M$ orientable covering of M. It is connected, and $\pi_1(\tilde{M}, \tilde{x}) \approx G$.

The orientable double covering is:

$$\tilde{M} = \{(x, \pm\mu_x) \mid x \in M, \mu_x \in H_n(M, M - x) \approx Z \text{ is a generator}\}$$

Put a topology on $\tilde{M}$ by $p : \tilde{M} \longrightarrow M$,

$$p(x, \pm\mu_x) = x, \qquad p^{-1}(B) = \underbrace{B \times \{-\mu_B\} \uplus B \times \{\mu_B\}}_{\text{2 open discs}}$$

Theorem: $\tilde{M}$ is connected if and only if M is not orientable.

Proof: Suppose M is orientable. Then there exists a section $s : M \longrightarrow \tilde{M}$ of the double covering $p : \tilde{M} \longrightarrow M$: $s(x) = (x, \mu_x)$ where μ_x is a consistent choice of orientations throughout M. $\square$

Corollary: M is orientable if there does not exist an epimorphism $\pi_1(M) \to \mathbb{Z}_2$.

The converse is false. Example: $M = S^1 \times S^1$ is orientable, yet there exists an epimorphism $\pi_1(M) \approx \mathbb{Z} \oplus \mathbb{Z} \twoheadrightarrow Z_2$.

14.3 Other ways of defining orientability

Recall the definition of a manifold M^n: a paracompact Hausdorff space ...etc ... $\{(U_i, V_i, \phi_i)\}_{i \in I}, U_i \subseteq M$ open. If $V_i \subseteq \mathbb{R}^n$ open, then $\phi_i : U_i \to V_i$ is a homeomorphism.

Definition: The manifold is *smooth* if

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \longrightarrow \phi_j(U_i \cap U_j)$$

is C^∞.

For a smooth manifold M^n we have a tangent bundle $TM \xrightarrow{p} M$, where $p : T_x(M) \longmapsto x \in M$. $T_x(M)$ is the tangent space of X at M, and $T(M) = \bigcup_{x \in M} T_x(M)$.

Definition: M^n is orientable if all of the transition functions $\phi_j \circ \phi_i^{-1}$ preserve orientation, i.e. the Jacobian determinant is always positive.

Locally: choose an ordered basis for $T_x(M)$, $e_1(x), \ldots, e_n(x)$. If $B \subseteq M$ is a disc $\approx \mathbb{R}^n$, then $T(B) \approx B \times \mathbb{R}^n$.

Definition: Two ordered bases $e_1(x), e_n(x)$ and $e'_1(x), \ldots, e'_n(x)$ of $T_x M \approx \mathbb{R}^n$ are *equivalent* if $\det L > 0$, where $L \in Gl_n(\mathbb{R})$ taking the e-basis to the $e'-$basis

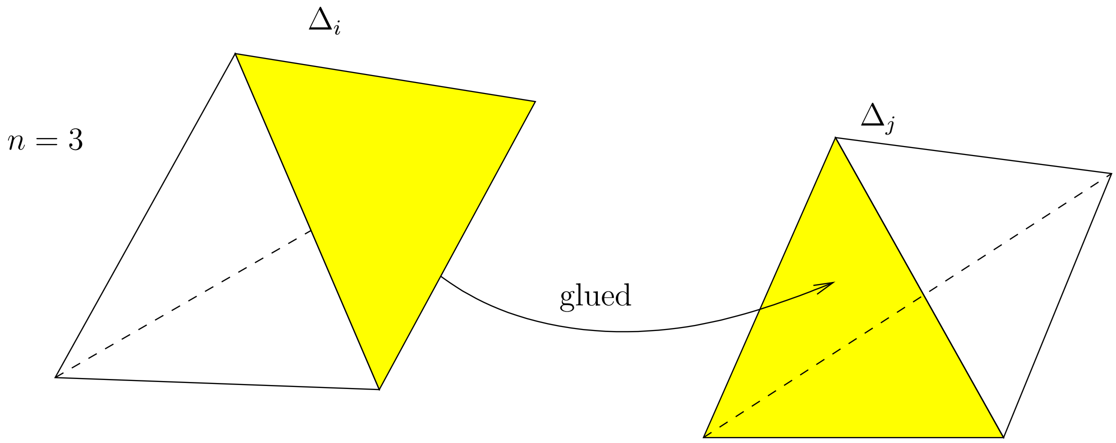
Remark: Since $T(B) \approx B \times \mathbb{R}^n$, we can choose equivalence classes of bases throughout B. That is, the bundle map $p : B \times \mathbb{R}^n \longrightarrow B$ has n linearly independent sections.

Remark: Now we can "analytically continue" an equivalence class of ordered bases at $x \in M$ along a path γ from x to y.

Theorem: Suppose M^n is a closed, connected n-manifold. Then

$$H_n(M; \mathbb{Z}) \approx \begin{cases} \mathbb{Z} & \text{if } M \text{ orientable} \\ 0 & \text{otherwise} \end{cases}$$

Proof: Assume that M is a simplicial complex. Let $\Delta_1, \Delta_2, \ldots, \Delta_k$ be the set of all n-simplexes in M. $M = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_k$.



For argument's sake, suppose $i = 1, j = 2$, so that $\Delta_1 \cap \Delta_2$ is some $n-1$-dimensional simplex. Assume that M is orientable.

Then there exists local orientations μ_Δ for each n-simplex Δ. Then the local orientations $\mu_{\Delta_1}, \mu_{\Delta_2}$ are compatible if and only if the common $n-1$-dimensional faces cancels in $\partial(\Delta_1 + \Delta_2)$ in the nth chain group.

Etc.

Therefore, a cycle in $C_n(M)$ is $\Delta_1 + \cdots + \Delta_n$.

Moreover, it then follows that a chain $c = n_1\Delta_1 + \cdots + n_k\Delta_k$ will be a cycle if $n_1 = n_2 = \cdots = n_k$. I.e. $H_n(M;\mathbb{Z})$ is generated by $[M] = \Delta_1 + \ldots + \Delta_k$, so $H_n(M) \approx \mathbb{Z}$. Here, M is called a *fundamental class* (the other one is $-M$).

Exercise: if M is not orientable, then the group of cycles in the nth chain group is zero (it is not possible to achieve cancellation as above). $\square$

Remark: Take coefficients in $R = \mathbb{Z}_2$. If Δ_1, Δ_2 have a common $n-1$-dimensional face Δ, then $\partial(\Delta_1 + \Delta_2) = \underbrace{2\Delta}_{=0} +$ some other $n-1$-simplexes.

Therefore, every closed, connected n-manifold is orientable with coefficients in $R = \mathbb{Z}_2$.

14.4 Poincaré Duality

Theorem: (Poincare Duality) Suppose that M^n is a closed, connected, orientable over R n-manifold. Let $[M]$ be a fundamental class. Then the homomorphism

$$H^i(M;R) \longrightarrow H_{n-i}(M;R)$$

$$u \longmapsto [M] \cap u$$

is an isomorphism for all i.

We do this locally, changing cohomology to cohomology with compact support. Then, write M as a union of contractible charts, and piece them together with the Meyer-Vietoris sequence. Proof omitted.

14.5 Reading: Chapter 4, Higher homotopy groups

Let X be a space with a base point x_0. Then we define the n^{th} homotopy group of X at x_0 by $\pi_n(X;x_0) = [(I^n, \partial I^n), (X, x_0)]$, where I^n is the n-cube, $I^n = \{(t_1, \ldots, t_n) \mid 0 \leq t_i \leq 1\}$.

If $n = 1$, then $\pi_1(X, x_0)$ is the fundamental group.

If $n = 0$ then I^0 is a point and $\partial I^0 = \emptyset$. Therefore Thus $\pi_0(X)$ is the set of path components of X. This isn't a group.

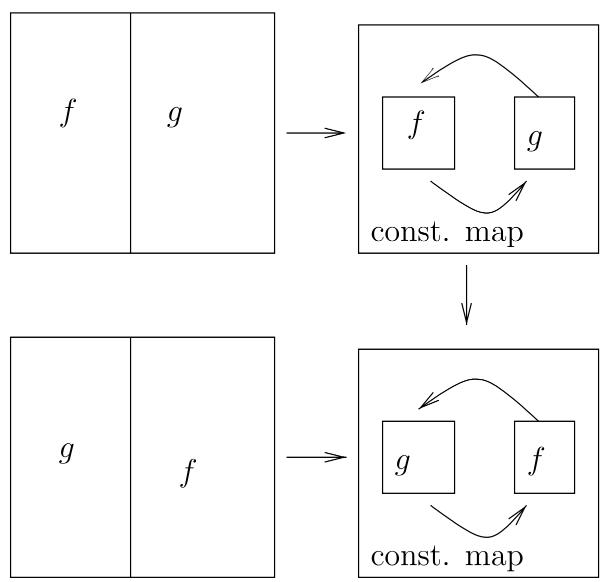
Remark:

1. $\pi_n(X, x_0)$ is functorial for maps $f : (X, x_0) \to (Y, y_0)$.
2. The suspension functor Σ and the loop space functor Ω are *adjoint*, i.e. $[\Sigma X, Y] \approx [X, \Omega Y]$. The mappings $f : \Sigma X \to Y$ and $g : X \to \Omega Y$

correspond if, and only if, $f(x,t) = g(x)(t)$. $[X, \Omega Y]$ is a group and therefore so is $[\Sigma X, Y]$.

3. $\pi_n(X, x_0) = [(S^n, s_0), (X, x_0)]$ since $I^n/\partial I^n = S^n$.

4. $\pi_n(X, x_0)$ is abelian for $n \geq 2$:



Chapter 15

March 3

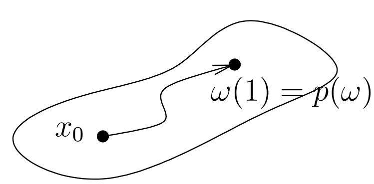
15.1 The path-loop fibration

Recall that ΩX the space of loops based at $x_0 \in X$, i.e. the space of all continuous maps $\omega : (I, \partial I) \to (X, x_0)$.

There is a canonical way to topologize a mapping space: it is the *compact open topology*. We give ΩX this topology.

Definition: The *path space based at* x_0, denoted PX, is the space of continuous functions $\omega : (I, 0) \to (X, x_0)$.

There exists a natural map $p : PX \to X$ defined by $p(\omega) = \omega(1)$. Then $\Omega = p^{-1}(x_0)$ and there is an inclusion $i : \Omega X \to PX$.



Comments:

- ΩX is an H-space – i.e. a group up to homotopy.
- PX is contractible.

Exercise: Find a homotopy $PX \times I \to PX$ that contracts PX.

ΩX is a topological group up to homotopy. That is, there exists a topological group GX and a homotopy equivalence $\Omega X \simeq GX$. This idea is due to John Milnor: 1956 Annals of Math., Universal bundles I and II.

Assume X is a countable simplicial complex (i.e. there are at most countably many simplices in each dimension). Then define $S_n(X)$ to be the space of all sequences $(x_n, \ldots, x_0)$, where $x_i \in X$, such that any two consecutive points x_i, x_{i+1} belong to a common simplex.

We can consider such a sequence as a rectilinear path from x_0 to x_n.

We give S_n the topology induced from the inclusion $S_n(X) \subseteq X^{n+1}$. Let $S(X) = \bigcup_{n \geq 0} S_n(X)$ and then define $E(X) = S(X)/\sim$, where the equivalence relation is defined by

$$(x_n, \ldots, x_i, \ldots, x_0) \sim (x_n, \hat{x}_i, x_0) \text{ if either } x_{i+1} = x_i \text{ or } x_{i+1} = x_{i-1}.$$

In terms of rectilinear paths this means we delete paths that are stationary at some x_i or paths that double back. Let $[x_n, \ldots, x_0]$ denote the equivalence class of $(x_n, \ldots, x_0)$. Then define a map $p : E(X) \to X$ by $p[x_n, \ldots, x_0] = x_n$. This map is continuous.

Let $G(X) = p^{-1}(x_0)$, the space of equivalence classes of rectilinear paths $[x_0, x_{n-1}, \ldots, x_0]$. GX is a topological group, where group multiplication is just concatenation of paths:

$$\begin{aligned}[x_0, x_{n-1}, x_0] \cdot [x_0, x'_{m-1}, \ldots, x'_1, x_0] &= [x_0, x_{n-1}, x_0, x_0, x'_{m-1}, \ldots, x'_1, x_0] \\ &= [x_0, x_{n-1}, x_0, x'_{m-1}, \ldots, x'_1, x_0]\end{aligned}$$

The identity is $[x_0] = [x_0, x_0] = [x_0, x_0, \cdots, x_0]$, and inverses are given by going backwards along a rectilinear path,

$$[x_0, x_{n-1}, \ldots, x_0]^{-1} = [x_0, x_1, \ldots, x_{n-1}, x_0].$$

Comments:

1. EX is contractible.
2. There exists a commutative diagram

$$\begin{array}{ccc} GX & \rightarrow & \Omega X \\ \downarrow & & \downarrow \\ EX & \rightarrow & PX \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

   where the maps $GX \to \Omega X$ and $EX \to PX$ are the natural inclusions.
3. GX acts on the right of EX by concatenation and $EX/GX \approx X$. In fact, $GX \to EX \to X$ is a principal universal GX-bundle. Therefore, the inclusion $GX \to \Omega X$ is a homotopy equivalence.

15.2 The James Reduced Product Construction JX

The James reduced product of a space X is $JX = \coprod_{k \geq 0} X^k / \sim$, where the equivalence relation is

$$(x_1, \ldots, x_j, \ldots, x_n) \sim (x_1, \ldots, \hat{x}_j, \ldots, x_n), \quad \text{if } x_j = e \text{ is the base point.}$$

Here, X^k is the k-fold cartesian power of X.

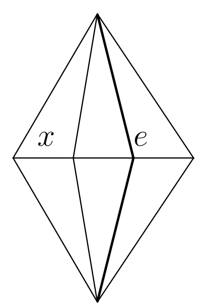
Let $J_m(X)$ be the subspace of $J(X)$ consisting of equivalence classes $[x_1, \ldots, x_n]$, $n \leq m$. For example, $J_1X = X$, $J_2(X) = X \times X/(x,e) \sim (e,x)$. So $J_1X \subset J_2X \subset J_3X \cdots \subset JX$.

JX has a multiplication $\mu : JX \times JX \to JX$ defined by concatenation, of equivalence classes. There is an identity, namely $[e]$, but there are no inverses.

Remark: JX is the free, associative H-space generated by X, taking e to be the identity. Think of the equivalence class $[x_1, \ldots, x_n]$ as the product $x_1x_2 \cdots x_n$.

Theorem: $JX \simeq \Omega\Sigma X$, where Σ is reduced suspension.

If $x \in X$, then x corresponds to a loop in the reduced suspension ΣX, defined by the following picture:



Therefore, we have a map $\gamma : J_1X \to \Omega\Sigma X$. The idea behind the proof of the theorem is to extend γ multiplicatively by: $\gamma[x_1, \ldots, x_n] = \gamma[x_1] \cdots \gamma[x_n]$. This extension is not continuous, but can be amended to work properly.

15.3 The Infinite Symmetric Product $SP^\infty(X)$

First we define the n^{th} symmetric product: $SP^n(X) := X^n/S_n$, where the action of the symmetric group S_n on X^n is by permuting coordinates. Thus $SP^n(X)$ is the space of unordered sequences of length n.

There are natural inclusions

$$SP^n(X) \subset SP^{n+1}(X), \{x_1, \ldots, x_n\} \to \{x_1, \ldots, x_n, e\}$$

Then we define the infinite symmetric product by

Definition: $SP^\infty(X) = \lim_{n\to\infty} SP^n(X) = \bigcup_{n\geq 0} SP^n(X)$.

Concatenation gives a commutative multiplication on $SP^\infty(X)$, with identity the base point e. $SP^\infty(X)$ is the free abelian associative H-space generated by X. We can think of $\{x_1, \ldots, x_n\}$ as the sum $x_1 + \cdots + x_n$.

Theorem: (Dold-Thom) $\pi_i(SP^\infty(X) \approx H_i(X)$ if $i \geq 1$.

Proof: The idea behing the proof is to show that the functor $X \to \{\pi_i(SP^\infty X)\}_{i\geq 1}$ satisfies the axioms for a reduced homology theory.

15.4 Higher Homotopy Groups

$$\pi_n(X, x_0) := [(I^n, \partial I^n), (X, x_0)] \approx [(S^n, s_0), (X, x_0)].$$

The most oustanding problem in algebraic topology in Algebraic Topology is to compute $\pi_n(X)$. This is an open, and very difficult, problem for $\pi_n(S^k)$.

Comment:

- Homology has "good" calculation tools. In particular, it has excision and Mayer-Vietoris sequence. Homotopy does not.
- If X is a CW complex of dimension n then $H_i(X) = 0$ when $i > n$, but this is not generally true for $\pi_i(X)$.

One of the calculation tools for higher homotopy is the following theorem.

Theorem: Suppose $p : \tilde{X} \to X$ is a covering of path connected spaces. Pick base points $x_0 \in X, \tilde{x}_0 \in p^{-1}(x_0)$. Then $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \xrightarrow{\approx} (X, x_0)$ for $n \geq 2$

Proof:

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\
(S^n, x_0) \arrow[r, "f"] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0)
\end{tikzcd}$$

Given any $f : (S^n, x_0) \to (X, x_0)$, there exists a unique lift $\tilde{f} : (S^n, x_0) \to (\tilde{X}, \tilde{x}_0)$. This follows from the unique path lifting property and the fact that $\pi_1(S^n) = 0$ if $n \geq 2$. $\square$

Examples.

$$\pi_i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}_2 & \text{if } i = 1 \\ 0 & \text{if } 1 < i < n \\ Z & \text{if } i = n \\ ? & \text{otherwise} \end{cases}$$

Proof: The universal covering space of $\mathbb{R}P^n$ is S^n if $n \geq 2$. Now use the theorem and the fact that $\pi_n(S^n) = \mathbb{Z}$. $\square$

Another fact is: if $f : S^i \to S^n$ and $i < n$, then $\exists$ a map $g : S^i \to S^n$ such that $f \simeq g : S^i \to S^n$ and Im g is contained in an i dimensional subcomplex of S^n. It follows that any such map is null homotopic.

Example: Let M_g be a surface of genus $g \geq 1$. Then a presentation for $\pi_1(M_g)$ is

$$\pi_1(M_g) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1 \rangle$$

where $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$.

Chapter 16

March 8

16.1 More on the Milnor Simplicial Path Loop Spaces

Recall the construction of EX and GX from the last chapter.

1. $GX \to EX \to X$ is a principal fiber bundle with group GX. There exists an action $EX \times GX \to EX$ which is concatenation of equivalence classes. The orbit space of this action is $EX/GX = X$.

2. $\Omega X \to PX \to X$ is a fibration (to be defined soon)

3. PX is contractible, and so is EX.

4. Any fibration (fiber bundle) $F \xrightarrow{i} E \xrightarrow{p} B$ has a long exact sequence in homotopy,

$$\cdots \to \pi_n(F,*) \xrightarrow{i_*} \pi_n(E,*) \xrightarrow{p_*} \pi_n(B,*) \xrightarrow{\partial} \pi_{n-1}(F,*) \to \cdots$$

5. The long exact sequence is natural for maps between fibrations:

$$
\begin{tikzcd}
0 \arrow[d, equal] & & & 0 \arrow[d, equal] \\
\pi_n(EX,*) \arrow[r, "p_*"] \arrow[d] & \pi_n(X,*) \arrow[r, "\partial", "\approx"'] \arrow[d] & \pi_{n-1}(GX,*) \arrow[r] \arrow[d] & \pi_{n-1}(EX,*) \arrow[d] \\
\pi_n(PX,*) \arrow[r, "q_*"] \arrow[d, equal] & \pi_n(X,*) \arrow[r, "\partial", "\approx"'] & \imath_{n-1}(\Omega X,*) \arrow[r] & \pi_{n-1}(EX,*) \arrow[d, equal] \\
0 & & & 0
\end{tikzcd}
$$

Therefore, $\pi_{n-1}(GX,*) \to \pi_{n-1}(\Omega X,*)$ is an isomorphism for all n.

Theorem: (Whitehead theorem) Suppose that X, Y are CW complexes, and $f : X \to Y$ is a base point preserving map such that $f_* : \pi_n(X,*) \to \pi_n(Y,*)$ is an isomorphism for all n. Then f is a homotopy equivalence.

Therefore, $GX \simeq \Omega X$.

Question: How does $\pi_n(X, x_0)$ depend on the base point x_0?

Remark: Suppose X is path connected. If γ is a path from x_0 to x_1 then there exists an isomorphism $h_{[\gamma]} : \pi_1(X, x_1) \xrightarrow{\approx} \pi_1(X, x_0)$, defined by

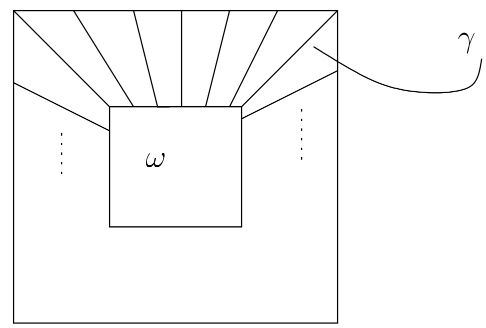
$$h_{[\gamma]}[\omega] := [\gamma * \omega * \gamma^{-1}].$$

If γ is a loop (i.e. $x_0 = x_1$ then $h_{[}\gamma]$ is conjugation by $[\gamma]$.

Remark: All of this applies to the higher homotopy groups $\pi_n(X,*)$. That is, if γ is a path from x_0 to x_1 then there exists an isomorphism $h_{[\gamma]} : \pi_n(X, x_1) \to \pi_n(X, x_0)$ for all n.

$n = 1$

$n = 2$



To put this another way, we have

Theorem: Suppose X is connected. Then $\pi_n(X, x_0)$ is a module over the group ring $\mathbb{Z}(\pi_1(X, x_0))$.

16.2 Relative Homotopy Groups

Suppose $x_0 \in A \subseteq X$. Then the relative homotopy groups $\pi_n(X, A, x_0)$ are defined by

$$\begin{aligned} \pi_n(X, A, x_0) &:= [(I^n, \partial I^n, J^{n-1}), (X, A, x_0)], \text{ where} \\ I^{n-1} &= \{(t_1, \dots, t_n) \in I^n \mid t_n = 0\} \\ J^{n-1} &= \partial I^n - I^{n-1} \end{aligned}$$

This is defined for $n \geq 1$. It is a group for $n \geq 2$, and is abelian if $n \geq 3$.

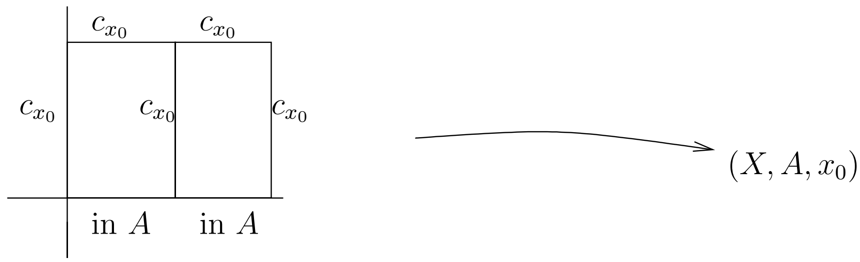
J^1 maps to x_0

$n = 2$ J^1 I^2 J^1 maps to x_0 f maps to x_0

I^1 maps to A

maps $f : (I^2, \partial I^2, J^1) \longrightarrow (X, A, x_0)$

Example: $\pi_2(X, A, x_0)$ is a group. To see this suppose we are given maps $f, g : (I^2, \partial I^2, J^1) \to (X, A, x_0)$. The following diagram indicates how to multiply the homotopy classes:



This also indicates how to define a group structure on $\pi_n(X, A, x_0)$.

Comments:

(a) π_n does not satisfy excision. That is, $\pi_n(X, A)$ is in general not isomorphic to $\pi_n(X - B, A - B)$.

(b) π_n does not have a Mayer-Vietoris sequence.

Therefore, the computation of the higher homotopy groups is "hard". However, we do have one computational tool:

Theorem: There exists a long exact sequence in homotopy.

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0)$$

Definition of the maps:

- $i_* : \pi_n(A, x_0) \to \pi_n(X, x_0)$ is the homomorphism induced by the inclusion $i : (A, x_0) \to (X, x_0)$.
- $j_* : \pi_n(X, x_0) \to \pi_n(X, A, x_0)$ is the homomorphism induced by the inclusion $j : (X, x_0, x_0) \to (X, A, x_0)$. Note that $\exists$ a natural isomorphism $\pi_n(X, x_0, x_0) \approx \pi_n(X, x_0)$.
- Let $f : (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ represent a homotopy class in $\pi_n(X, A, x_0)$. Then $\partial[f] = [f|S^{n-1}]$.

Example: $\pi_n(CX, X, x_0) \approx \pi_{n-1}(X, x_0)$, where CX is the cone of X.

Proof: Consider part of the long exact sequence in homotopy for the pair (CX, X) :

$$\begin{array}{ccccccc}
\pi_n(CX, x_0) & \to & \pi_n(CX, X, x_0) & \xrightarrow{\partial} & \pi_{n-1}(X, x_0) & \to & \pi_{n-1}(CX, x_0) \\
|| & & & & & & || \\
0 & & & & & & 0
\end{array}$$

Therefore $\partial : \pi_n(CX, X, x_0) \xrightarrow{\approx} \pi_{n-1}(X, x_0)$.

Definition: A map $p : E \to B$ has the *homotopy lifting property* with respect to a space Y if given any commutative diagram

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{h}"] \arrow[d, hook] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "H"] & B \end{tikzcd}$$

there exists a homotpy $\tilde{H} : Y \times I \to E$ lifting H.

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{h}"] \arrow[d, hook] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "H"] \arrow[ur, "\tilde{H}"] & B \end{tikzcd}$$

Remark: Suppose $h_0 \simeq h_1 : Y \to B$ are homotopic maps. If we can lift h_0 to E, then we can also lift h_1. In fact we can lift the entire homotopy.

Definition: A mapping $p : E \to B$ is a *fibration* if it has the lifting property for all spaces Y.

Definition: A mapping $p : E \to B$ is a *Serre fibration* if it has the lifting property for all discs D^n.

Theorem: Let $p : E \to B$ be a fibration. Choose base points $b_0 \in B$ and $e_0 \in p^{-1}(b_0) \subset E$, and let $F = p^{-1}(b_0)$ be the full image of the base point. Then p induces an isomorphism $p_* : \pi_n(E, F, b_0) \xrightarrow{\approx} \pi_n(B, b_0)$.

Corollary: We have a long exact sequence

$$\cdots \to \pi_n(F, *) \xrightarrow{i_*} \pi_n(E, *) \xrightarrow{p_*} \pi_n(B, *) \xrightarrow{\partial} \pi_{n-1}(F, *) \to \ldots \to \pi_0(B)$$

Proof: Apply the above Theorem to the long exact sequence in homotopy for the pair (F, E). $\square$