6. Logarithms

Last time, we tried to solve the equation \( e^x = 8 \). But, the problem was that "we do not know how to write 8 as \( e^x \)! In other words, there is no power of "e" that gives us 8.

\[ e^x = 8 \Rightarrow \text{we cannot solve it} \]

In order to re-solve this problem, we need to define a new function which is called \textbf{Logarithms}.

\[ e^x = 8 \Rightarrow x = \log_e 8 \]

let go back to \( 2^x = 8 \), so,

\[ 2^x = 8 = 2^3 \Rightarrow x = 3 \rightarrow \text{we did before} \]

Now, we use log function to write

\[ 2^x = 8 \Rightarrow x = \log_2 8 = 3 \]

"what we think": what should we take as the power of 2 to give us 8.

\[ 2^0 = 8 \Rightarrow 0 = 3. \]
In general, if \( x = b^y \), then \( y = \log_b x \).

\[ 2^y = 4 \implies y = \log_2 4 = \log_2 2^2 = 2 \]

Go from \( x = b^y \) to \( y = \log_b x \) and vice versa!

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Example 1: compute

(A) \( y = \log_3 9 \). We can write \( 9 = 3^2 \). So, we have

\[
y = \log_3 9 = \log_3 3^2 = 2
\]

Now, we’re looking for \( 3^2 = 9 \). This means that \( 3^0 = 1 \). So,

\[
y = \log_3 9 = 2
\]
(B) \( y = \log_2(16) \). With the same idea, we first write
\[
16 = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4. \text{ So,}
\]
\[
y = \log_2(16) = \log_2(2^4) = 4\]
\[
2^4 = 16
\]

**Definition**: \( f(x) = \log_b x \) is called the logarithms function and it is defined as the inverse function of \( g(x) = b^x \).

- **Domain** \( g(x) = (-\infty, \infty) \)
- **Range** \( g(x) = (a, \infty) \)
- **x-axis**
- **y-axis**

"Two inverse functions are symmetry respect to the line \( y = x \)."

**Note**: Domain of \( g(m) = \text{Range of } f(m) \)
Range of \( g(m) = \text{Domain of } f(m) \)

\( f(m) \) and \( g(m) \) are inverse of each other.
6.1. Natural Logarithm

As we saw earlier, the case when the base \( b = e \) is special \( \Rightarrow \) remember \( y = e^x \)!

For log function, when \( b = e \) is especially important for us:

\[
\log_e x = \ln x
\]

So, the functions \( e^x \) and \( \ln x \) are inverse of each other and we have two important formulas:

1. \( \ln e = 1 \)
2. \( e^{\ln x} = x \)

Example 2: Compute \( \ln(e) \) and \( \ln(\sqrt{e}) \).

- \( \ln e = \log_e e = 0 \)
- \( e = e \Rightarrow \ln e = 1 \)
- \( \ln \sqrt{e} = \ln e^{\frac{1}{2}} = \log_e e^{\frac{1}{2}} = 0 \)
- \( e = e^{\frac{1}{2}} \Rightarrow \ln \sqrt{e} = \frac{1}{2} \)

\( \Rightarrow \ln \sqrt{e} = \frac{1}{2} \)