You'll be asked to fill out the table.

- Quiz #1: Memorize sin & cos of the special angles.
- Homework 2: Question 3 has been changed.
- Lab 2: Question 1 has been changed.

"Homework" ↦ Do not be shy.

* Please email me if you have any questions about HW & Lab.

5. Exponential Functions

Exponential functions have the variable \( x \) (or any other independent variable) in the exponent. Let's compare "polynomials" and "exponentials":

- \( x^2 \) and \( 2^x \)
- \( x^3 \) and \( 3^x \)
- \( a^4 \) and \( a^x \)

In general, if \( b \) is a real number (a constant), then

\[
f(x) = b^x
\]

is an exponential function.

Dumb cases:

- \( b = 0 \) → undefined
- \( f(x) = 0^x \) → 0
- \( b = 1 \) → \( f(x) = 1^x = 1 \).
We also consider the cases where the base "b" is positive. Therefore,

\[ f(x) = b^x, \quad b > 0 \]
\[ b \neq 0, \quad b \neq 1 \]

Let's think about the case when \( b = 2 \). So, we have

\[ f(x) = 2^x. \]

Now, if \( x = a \) is a natural number (1, 2, 3, 4, ...), we observe the following:

1. \[ 2^a = \underbrace{2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2}_{a \text{ times}} \]
2. \[ 2^a \cdot 2^b = \underbrace{2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2}_{a \text{ times}} \underbrace{2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2}_{b \text{ times}} \]

\[ = \underbrace{2 \cdot 2 \cdot 2 \cdot \ldots \cdot 2}_{a+b \text{ times}} \]
\[ = 2^{a+b} \]

\[ \Rightarrow 2^a \cdot 2^b = 2^{a+b}. \]
3. \((2^a)^b = (2^a \cdot 2^a \cdot 2^a \cdots \cdot 2^a)\)  
   \[= \underbrace{(2 \cdot 2 \cdots 2)}_{a \times} \underbrace{(2 \cdot 2 \cdots 2)}_{a \times} \cdots \underbrace{(2 \cdot 2 \cdots 2)}_{a \times} \]  
   \[= 2^{a \cdot b}\]  

\[\text{So,} \quad (2a)^b = 2^{a \cdot b}\]

What is \(2^{-1}\)?  
\[\begin{align*} 2^{-1} &= \frac{1}{2} \\ &= \frac{1}{2^1} \end{align*}\]

What is \(2^{-2}\)?  
\[\begin{align*} 2^{-2} &= \frac{1}{2^2} \\ &= \frac{1}{4} \end{align*}\]

4. \[2^{-a} = \frac{1}{2^a}\]

What is \(2^{\frac{1}{2}}\)?  
\[2^{\frac{1}{2}} = \sqrt{2} = \sqrt{2}\]

What is \(2^{\frac{1}{3}}\)?  
\[2^{\frac{1}{3}} = \sqrt[3]{2} = \sqrt[3]{2}\]

5. \[\frac{1}{a} = \sqrt[2]{2}\]

6. \(2^{\frac{a}{b}} = ?\)  
\[\Rightarrow (2^{\frac{1}{2}})^a = (\sqrt{2})^a\]

\[\Rightarrow (2^a)^{\frac{1}{b}} = b\sqrt{2^a}\]

\[\Rightarrow 2^{\frac{a}{b}} = b\sqrt{2^a}\]

7. \[\frac{2^a}{2^b} = 2^a \cdot 2^{-b} = 2^{a-b}\]

So, \[\frac{2^a}{2^b} = 2^{a-b}\]
Let's plot \( f(x) = 2^x \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) = 2^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

What about \( f(x) = 3^x \)?

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f(x) = 3^x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

\[ f(x) = b^x \quad \text{\( b = e \)} \]

\[ f(x) = e^x \quad \Rightarrow \quad \text{Natural exponential function} \]

This function appears in many natural phenomena, and in general, we just call it the exponential function.

\* What is \( e \)? \( 2 < e < 3 \)

\( e \) is a math constant, and it is approximately

\[ 2 < e = 2.71828182 \ldots < 3 \]
Important Note

For any natural numbers \( a \) and \( b \):

1. \( e^a \cdot e^b = e^{a+b} \)
2. \((e^a)^b = e^{ab} = (e^b)^a \)
3. \( e^{-a} = \frac{1}{e^a} \)
4. \( \frac{e^a}{e^b} = e^{a-b} \)
5. \( e^{a/b} = (\sqrt[b]{e})^a = \sqrt[e^a]{e^b} \)
6. \( e^{\frac{1}{a}} = \sqrt[a]{e} \).

Example 1: Simplify \( (\sqrt{e} \cdot e^2)^3 = ? \)

Nominator:

\[ (\sqrt{e} \cdot e^2)^3 = (e^{\frac{1}{2}} \cdot e^2)^3 = (e^{\frac{5}{2}})^3 = e^{\frac{15}{2}} \cdot 3 = e^{\frac{15}{2}} \cdot e = e^{\frac{17}{2}} \]

\( \frac{e^{\frac{17}{2}}}{e} = e^{\frac{15}{2}} \)

So, we have \( e^{\frac{15}{2}} \).

Example 2: Simplify \( e^{x^3} \cdot e^{2x} \cdot e^x = ? \)

Nominator:

\[ e^{x^3} \cdot e^{2x} \cdot e^x = (e^{x^3}) \cdot (e^{2x}) \cdot e^x = e^{x^3+2x} \cdot e^x = e^{x^3+3x} \]
\[
\frac{e^x \cdot 2^n}{e^3} = \frac{e^x + 3n}{e^3} \quad (4) \quad \frac{3^n}{x + 3n - 3}
\]

Let's try to solve equations with exponentials. For example, take a look at \(2^x = 8\), we can find \(x\) that satisfies this equation in the following way:

We can write \(8 = 2^3\), so,

\[2^x = 8 = 2^3 \quad \Rightarrow \quad 2^x = 2^3 \quad \Rightarrow \quad \boxed{x = 3}\]

What about \(e^x = 1\)?

We can write \(1 = e^0\), so, we have

\[e^x = 1 = e^0 \Rightarrow e^x = e^0 \Rightarrow \boxed{x = 0}\]

But, what about \(e^x = 8\)?

Can we write \(8 = e^x\)? \(\square\) \(\Rightarrow\) ? \(\boxed{\text{No}}\)

\[\text{That's why we need Logarithms.}\]