Lecture note 23 (Nov 1, 2017)

- No Lab this week (Oct 31 and Nov 2)
- Lab #8 next week (Nov 7 and Nov 9)
- Homework #6 will be posted later Today!

Integration

The tangent lines to curves \(\rightarrow\) Derivative
the area underneath curves \(\rightarrow\) Integral

1. Approximating Areas by Riemann Sums:
   we first recall that

\[
\begin{align*}
\text{Area} &= \frac{a \cdot b}{2} \\
\text{Area} &= a \cdot b \\
\text{Area} &= a^2 \\
\text{Area} &= r^2 \pi
\end{align*}
\]
But how can we compute the area of the following:

\[ y = f(x) \]

What is our goal and idea?

I. Approximate the area using "Rectangles!"

II. Better and better approximations to get smaller "Error"!

Today, we focus on (I) and by solving examples we'll see how we can reach out (II). Next lecture, we mainly talk about (II).

Let's consider \( f(x) = x^2 \) and its graph between \( x = 1 \) and \( x = 3 \):
Let's use two rectangles to estimate the area:

\[ n = 2 \rightarrow \text{number of rectangles} \]

\[ y = x^2 = f(x) \]

Exact area \( = 8.666... \approx 8.67 \)

we'll consider two different cases:

- **Right endpoints**:

  \[ f(3) = 9 \]
  \[ f(2) = 4 \]
  \[ f(1) = 1 \]

- **Left endpoint**:

  \[ f(3) = 9 \]
  \[ f(2) = 4 \]
  \[ f(1) = 1 \]

\[ \text{height} \rightarrow \text{area} = \text{height} \times \text{base} \]

\[ \text{Area} = A_1 + A_2 = (1)(4) + (1)(9) = 13 \]

\[ \text{Area} = (1)(1) + (1)(4) = 5 \]
So, the exact area has to be between 5 and 13. We know 8.67 is the exact area!

What is the problem? Not good enough approximation.

How to fix it? and more and more

Use more rectangles...

Example 1: Consider the region bounded by the graph of \( f(x) = x^2 \) between \( x = 1 \) and \( x = 3 \). Estimate the area using four approximating rectangles and

(i) Right endpoints \( R_4 \)

\[
\begin{align*}
\Delta x &= \frac{b-a}{n} \\
\Delta x &= \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \\
\Delta x &= 0.5
\end{align*}
\]

\[
\begin{align*}
a &= 1 \\
b &= 3
\end{align*}
\]
\[
\text{Area} = A_1 + A_2 + A_3 + A_4 \\
= (0.5)(2.25) + (0.5)(4) + (0.5)(6.25) \\
+ (0.5)(8.9) = 10.75 \rightarrow \text{much better than } 13 \\
\text{closer to } 8.67 \\
\text{(ii) Left endpoints} (L_n) \\
\text{exact area} \\
\text{from } n=2 \\
\]

\[
\text{Area} = A_1 + A_2 + A_3 + A_4 \\
= (0.5)(1) + (0.5)(2.25) + (0.5)(4) \\
+ (0.5)(6.25) \\
= 6.75 \rightarrow \text{much better than 5} \\
\text{from } n=2 \\
\]
In general, for any number \( n \), we can split the interval \([a, b]\) to \( n \) sub-intervals of equal length \( \Delta x = \frac{b-a}{n} \) with \( a = x_0 \) and \( b = x_n \).

\[
\begin{align*}
\Delta x &\quad \Delta x \\
x_0 &= a & x_1 & x_2 & x_3 & x_4 & x_5 & \cdots & x_{n-4} & x_{n-3} & x_{n-2} & x_{n-1} & b = x_n
\end{align*}
\]

"Riemman Sum": Given function \( f(x) \), the following is called Riemman Sum:

\[
\text{Area} = f(x_1^*) \Delta x + f(x_2^*) \Delta x + \cdots + f(x_n^*) \Delta x
\]

- \( x_1^*, x_2^*, \ldots, x_n^* \rightarrow \text{left endpoints} \)

- \( x_1^*, x_2^*, \ldots, x_n^* \rightarrow \text{right endpoints} \)

Riemman Sum